Game-Theoretic Reversibility and $m$-Valued Quantum Computation

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ABSTRACT

This paper introduces the new concept of Reversible Game Theory (RGT) and its Multiple-Valued Quantum Computation (MVQC). Game Theory (GT) involves the study of competition and cooperation, without regard to the particular entities (agents) involved, and issues of rationality associated with such phenomena. Reversibility property in GT can be important in situations where: (1) an outside observer needs to know reversely the paths that lead to specific payoffs on a game’s extensive form, and (2) modeling the (maximin) dynamics using low-power consuming circuits as reversibility is a main requirement for low-power circuit design of future technologies such as in Quantum Computing (QC). Quantum Decision Trees (QDTs) are also presented as a quantum representation for applying MVQC to games’ dynamics.

Keywords: Circuits and Systems, Game Theory, Quantum Circuits, Quantum Computing, Reversible Logic, Reversible Circuits, Low-Power Computing, Low-Power VLSI Circuit Design.

1. INTRODUCTION

The notion of implementing Game Theory (GT) using quantum computing has been suggested by various authors, e.g., (Benjamin and Hayden, 2001; Johnson, 2001; Meyer, 1999). Researchers have succeeded in implementing many-valued ($m$-valued) logic gates using many-valued quantum systems, e.g., (Muthukrishnan and Stroud, 2000). This paper investigates the issues of reversibility and the implementation of games’ maximin dynamics using Many-Valued Quantum Computing (MVQC).

Quantum Computing (QC) is a method of computation that uses a dynamic process governed by the Schrödinger Equation (SE) (Al-Rabadi, 2004; Nielsen and Chuang, 2000). Research in QC gained momentum, for its application within the context of GT, e.g., (Kay, Benjamin and Johnson, 2001; Peterson, 1999; Piotrowski and Sladkowski, 2002), when it has been shown that the Prisoner’s Dilemma (PD) game, which is not solvable in a single iteration and can be classically solved in several ways (Luce and Raiffa, 1989; Zagare, 1984) such as 1. iterated games, 2. non-myopic rationality, and 3. meta-games, can also be solved using Quantum Computing (QC) method (Eisert, Wilkens and Lewenstein, 1999).

Other motivations for pursuing the possibility of implementing game dynamics using Reversible Logic (RL) and QC would include items such as: 1. power: the fact that, theoretically, the internal computations in RL systems consume no power. It is shown in (Landauer, 1961) that a necessary (but not sufficient) condition for not dissipating power in any physical circuit, is that all system circuits must be built using fully reversible logical components. For this reason, different technologies have been studied to implement reversible logic in hardware, such as adiabatic CMOS (Roy and Prasad, 2000), optical (Picton, 1991) and quantum (Al-Rabadi, 2004; Muthukrishnan and Stroud, 2000; Nielsen and Chuang, 2000). Fully reversible digital systems will greatly reduce the power consumption (theoretically eliminate) through three conditions: (i) logical reversibility: the vector of input states can always be uniquely reconstructed from the vector of output states, (ii) physical reversibility: the physical switch operates backwards as well as forwards, and (iii) the use of “ideal-like” switches that have no parasitic resistances; 2. size: since the newly emerging quantum computing technology must be reversible (Al-Rabadi, 2004; Bennett, 1973; Nielsen and Chuang, 2000), the current trends related to more dense hardware
implementations are heading towards 1 Angstrom (atomic size), at which quantum mechanical effects have to be accounted for; and 3. speed: if the properties of superposition and entanglement of quantum mechanics can be usefully employed in the GT context, significant computational and modeling speed enhancements can be expected (Al-Rabadi, 2004; Nielsen and Chuang, 2000).

A main objective of this paper is to introduce two aspects in GT: 1. reversibility, and 2. Multiple-Valued Quantum Computation (MVQC) of games’ (maximin) dynamics.

Basic background is given in Section (2). Logic reversibility in game theory is presented in Section (3). The implementation of games using multiple-valued (m-valued) quantum computing is introduced in Section (4). Conclusions and future work are presented in Section (5).

2. FUNDAMENTALS

This Section presents basic background in the topics of reversible logic, game theory, and quantum computing.

2.1. REVERSIBLE LOGIC

An \((n, k)\) reversible circuit is a circuit that has \(n\) number of inputs and \(k\) number of outputs and is one-to-one mapping between vectors of inputs and outputs, thus, the vector of input states can be always uniquely reconstructed from the vector of output states (Al-Rabadi, 2004; Bennett, 1973; DeVos, 1999; Landauer, 1961; Nielsen and Chuang, 2000; Picton, 1991; Roy and Prasad, 2000; Shi and Lee, 2000). The auxiliary outputs that are needed only for the purpose of reversibility are called “garbage” outputs. These are auxiliary outputs from which a reversible map is constructed (cf. Example (1)).

An algorithm called Reversible Boolean Function (RevBF) that produces a reversible form from an irreversible Boolean function is as follows (Al-Rabadi, 2004). (We assume the Boolean function is specified via a table, as in Example (1)).

**Algorithm RevBF**
1. Add a sufficient number of auxiliary output variables such that, the number of outputs equals the number of inputs. Allocate a new column in the mapping’s table for each auxiliary variable.
2. For construction of the first auxiliary output, assign a constant \(C_1\) to half of the cells in the corresponding table column (e.g., zeros), and the second half as another constant \(C_2\) (e.g., ones). For convenience, one may assign \(C_1\) to the first half of the column, and \(C_2\) to the second half of the column (cf. Table 1, column \(Y_1\)).
3. For the next auxiliary output, if non-reversibility still exists, then assign for identical output tuples (irreversible map entries) values which are half zeros and half ones (cf. Table (1), first two entries of column \(Y_2\)), and then assign a constant for the remainder that are already reversible (cf. bottom two entries of \(Y_2\)).
4. Do step 3, until all map entries are reversible.

**Example 1.** The standard two-variable Boolean AND: \(Y = X_1 \cdot X_2\) is irreversible. The following table lists the mapping components:

<table>
<thead>
<tr>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Applying the above RevBF algorithm, the following is one possible reversible two Boolean variables AND:

**Table (1): (2, 3) Reversible map for Boolean AND.**

<table>
<thead>
<tr>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(Y)</th>
<th>(Y_1)</th>
<th>(Y_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
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<td>0</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Using the RevBF algorithm, the construction of the reversible map in Table (1) is obtained as follows: since \(Y\) is irreversible, assign auxiliary (“garbage”) output \(Y_1\) and assign the first half of its values the constant “0”, and the second half another constant “1”. Since the new map is still irreversible, assign a new garbage output \(Y_2\) and assign the first cell value to “0” and the second cell value to “1”.

2.2. GAME THEORY

Game Theory (GT) is an important modeling method used in several areas like economics, business, finance, marketing, biology, engineering, anthropology, social psychology, politics and philosophy (Axelrod, 1984; Barry and Hardin, 1982; Davis, 1997; Glance and Huberman, 1994; Hamburger, 1979; Hardin, 1968; Luce and Raiffa, 1989; Neumann and Morgenstern, 1953;

While the context of Decision Theory (DT) is one player games against nature, the context of GT is the decision making and moves based on payoffs (utilities), for two players or more than two players (Hamburger, 1979; Luce and Raiffa, 1989; Neumann and Morgenstern, 1953; Zagare, 1984). Game theory deals with situations that are: 1. pure conflict (strictly competitive; Zero Sum Games (ZSG)) where what one player wins is equal to what the other player loses and the total pay (utility) is constant, 2. partial conflict (competition and cooperation; Non-Zero Sum Games (NZSG)), where the total pay (utility) is variable and 3. pure cooperation. While a ZSG has a theory that leads to a solution, a NZSG has no general solution and often exhibits paradoxical features (Hamburger, 1979; Luce and Raiffa, 1989; Zagare, 1984).

Problem representation in GT can be ordinal utility versus interval (cardinal) utility, and extensive (tree-based) form where time is explicit versus normal (matrix; table) form, using a payoff matrix (table), where time is only implicit. Typically, some information is lost by going from the extensive form to the normal form. Information in GT can be: 1. perfect information that means that one knows always where he is on tree in the extensive form, 2. complete information where the payoff table is known, or 3. incomplete information where the payoff table is not known. Principle of rationality in GT states that 1. one maximizes (or secures some minimum) utility by some appropriate decision rule, and 2. one assumes the other player is doing likewise and acts accordingly for one’s own utility.

A strategy in GT is a decision or a sequence of decisions, and a dominant strategy is a strategy that is optimal no matter what the other player does. A strategy that solves a game without probabilistic choices is called pure strategy, and a strategy that uses probability to solve a game is called mixed strategy, i.e., if a player in a game chooses among two or more strategies at random according to specific probabilities, then this choice is called a "mixed strategy." If a pure strategy fails to solve the game, then one uses mixed strategy.

A players’ solution to a ZSG can be a maximin: get the best of the worst possible outcomes, where a dominant strategy is automatically maximin. A joint (mutual) maximin means that both the row and column players use maximin. In ZSG, if the column (row) player does maximin on his own payoffs, he will choose the same strategy as if he does minimax on the row (column) players’ payoffs (while the value (outcome payoff) would be different) (Hamburger, 1979; Luce and Raiffa, 1989; Neumann and Morgenstern, 1953; Zagare, 1984).

A Saddle Point (SP) is a cell which is the maximum utility in its column and minimum utility in its row and is a cell, which neither the row player nor the column player wants to move from. An Equilibrium Point (EP) is the cell(s) that game dynamics settle in. In ZSG, saddle points are the same as equilibrium points, but this is not necessarily true in a NZSG. Pareto Optimal (PO) cell(s) is the cell that makes everyone happy and moving from it will hurt at least one player, i.e., is a cell that you remain at, if any player vetoes (all) movements to (all) other cells (or in other words it is the cell that moving away from will hurt at least one player; the cell that, if both players would be willing to move to, is identical to the original cell). A Non-Pareto Optimal (NPO) cell is a cell that a player moves from since no player vetoes that movement to another cell(s), i.e., if you can move to at least one another cell then the current cell is NPO.

Hierarchy of 2-player ZSGs can be: 1. if both players have dominant strategy then each player uses dominant strategy; 2. if one player has dominant strategy then one uses the dominant strategy and the other acts appropriately; 3. if no player has dominant strategy but the game has a saddle point then both players do pure maximin (and in this case maximin equals minimax); and 4. if the game has no saddle point then both players do mixed maximin (Luce and Raiffa, 1989; Zagare, 1984).

In ZSG, a game is called unstable if the joint (mutual) maximin cell is not the same as EP cell, otherwise the game is stable. Figure (1) illustrates common games in GT where a lighter color means maximin, a darker color means EPs, po is Pareto optimal, and npo is non-Pareto optimal.

2.3. QUANTUM COMPUTING

QC is a method of computation that uses a dynamic process governed by the Schrödinger Equation (SE) (Al-Rabadi, 2004; Nielsen and Chuang, 2000). The one-dimensional Time-Dependent SE (TDSE) takes the following general form:

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) = i\hbar\frac{\partial}{\partial t}$$

(1)
amplitudes
Hamiltonian operator (transform an input vector of quantum bits (qubits) into an output vector of qubits (Al-Rabadi, 2004; Nielsen and Chuang, 2000).

Figure 1. Games (going left-to-right and top-to-down): (a) ordinal-utility pure-strategy chicken game, (b) ordinal-utility pure-strategy Prisoner’s Dilemma (PD) game, (c) ordinal-utility pure-strategy assurance game, (d) interval-utility mixed-strategy leader (battle of the sexes) game, (e) ordinal-utility pure-strategy hero game, and (f) ordinal-utility pure-strategy convergence game.

or: \( H|\psi\rangle = i(h/2\pi) \frac{\partial|\psi\rangle}{\partial t} \) (2)

where \( h \) is Planck’s constant (6.626 \times 10^{-34} \text{ J-s}), \( V(x,t) \) is the potential, \( m \) is particle’s mass, \( i \) is the imaginary number, \( |\psi(x,t)\rangle \) is the quantum state, \( H \) is the Hamiltonian operator \( (H = -(h/2\pi)^2/2m)\nabla^2 + V) \), and \( \nabla^2 \) is the Laplacian operator.

While the above holds for all physical systems, in the Quantum Computing (QC) context, the Time-Independent SE (TISE) is normally used (Al-Rabadi, 2004; Nielsen and Chuang, 2000):

\( \nabla^2 |\psi\rangle = \frac{-2m}{(h/2\pi)^2} (V-E)|\psi\rangle \) (3)

where the solution \( |\psi\rangle \) is an expansion over orthogonal basis states \( |\phi\rangle \) defined in Hilbert space \( \mathbb{H} \) as follows:

\[ |\psi\rangle = \sum_i c_i |\phi_i\rangle \] (4)

where the coefficients \( c_i \) are called probability amplitudes, and \( |c_i|^2 \) is the probability that the quantum state \( |\psi\rangle \) will collapse into the (eigen) state \( |\phi_i\rangle \). The probability is equal to the inner product \( |<\phi_i |\psi\rangle|^2 \), with the unitary condition \( \sum_i |c_i|^2 = 1 \).

In QC, a linear and unitary operator \( \mathcal{U} \) is used to transform an input vector of quantum bits (qubits) into an output vector of qubits (Al-Rabadi, 2004; Nielsen and Chuang, 2000).

In two-valued QC, a qubit is a vector of bits defined as follows (Al-Rabadi, 2004; Nielsen and Chuang, 2000):

\[ \text{qubit}_0 = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{qubit}_1 = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \] (5)

A two-valued quantum state \( |\psi\rangle \) is a superposition of quantum basis states \( |\phi\rangle \), such as those defined in Equation (5). Thus, for the orthonormal computational basis states \( |0\rangle, |1\rangle \), one has the following quantum state:

\[ |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \] (6)

where \( \alpha^2 + |\alpha|^2 = p_0 = \) the probability of having state \( |\psi\rangle \) in state \( |0\rangle \), \( |\beta|^2 = |\beta|^2 = p_1 = \) the probability of having state \( |\psi\rangle \) in state \( |1\rangle \), and \( |\alpha|^2 + |\beta|^2 = 1 \). The calculation in QC for multiple systems (e.g., the equivalent of a register) follow the tensor product (\( \otimes \)) (Al-Rabadi, 2004; Nielsen and Chuang, 2000). For example, given two states \( |\psi_1\rangle \) and \( |\psi_2\rangle \) one has the following QC:

\[ |\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = (|\psi_1\rangle \otimes |\psi_2\rangle) = \left( \alpha_1 |0\rangle + \beta_1 |1\rangle \right) \otimes \left( \alpha_2 |0\rangle + \beta_2 |1\rangle \right) = \alpha_1 \alpha_2 |00\rangle + \alpha_1 \beta_2 |01\rangle + \beta_1 \alpha_2 |10\rangle + \beta_1 \beta_2 |11\rangle \] (7)

A physical system, describable by the following equation (Al-Rabadi, 2004; Nielsen and Chuang, 2000):

\[ |\psi\rangle = c_1 |\text{Spinup}\rangle + c_2 |\text{Spindown}\rangle \] (8)

(e.g., the hydrogen atom), can be used to physically implement a two-valued QC. Another common alternative form of Equation (8) is:

\[ |\psi\rangle = c_1 \frac{1}{\sqrt{2}} + c_2 \frac{1}{\sqrt{2}} \] (9)

Many-Valued QC (MVQC) can also be accomplished (Muthukrishnan and Stroud, 2000). For the three-valued QC, the \textit{qubit} becomes a 3-dimensional vector \textit{qudit}, and in general, for MVQC the qudit is of dimension “many”. For example, one has for 3-state QC (in Hilbert space \( \mathbb{H} \))
the following qudits:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

A three-valued quantum state is a superposition of three quantum orthonormal basis states (vectors). Thus, for the orthonormal computational basis states \(|0>, |1>, |2>\), one has the following quantum state:

\[
|\psi\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle
\]

where \(|\alpha|^2 = p_0 = \text{probability of having state } |0\rangle, |\beta|^2 = p_1 = \text{probability of having state } |1\rangle, |\gamma|^2 = p_2 = \text{probability of having state } |2\rangle,\) and \(|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1\).

In general, for an \(n\)-valued logic, a quantum state is a superposition of \(n\) quantum orthonormal basis states (vectors). Thus, for the orthonormal computational basis states \(|0>, |1>,...,|n-1\rangle\), one has the following quantum state:

\[
|\psi\rangle = \sum_{k=0}^{n-1} c_k |q_k\rangle
\]

where \(\sum_{k=0}^{n-1} c_k = 1\).

The calculation in QC for many-valued multiple systems follow the tensor product in a manner similar to the one demonstrated for two-valued QC in Equation (7). (The Appendix illustrates an example of implementing MVQC by exposing a particle to a potential field \(V\) where the distinct energy states are used as the orthonormal basis states.)

3. REVERSIBLE GAME THEORY (RGT)

In game theory, decision path reversibility can be of fundamental importance in cases where a third neutral party needs to know the temporal path that leads to specific outcomes between two competitive parties. We define the following: 1. the column and row players’ choices as inputs, 2. the corresponding utilities (payoffs) as the outputs, and 3. the path of player’s decisions (choices) to specific payoffs (utilities) as a decision path. A game is called logically irreversible if an outside observer cannot reconstruct the input states from the output states, otherwise the game is called logically reversible.

To solve the problem of irreversibility in GT, i.e., to produce a reversible extensive form of the game from its irreversible counterpart, one can use the following new algorithm called Reversible Decision Game Theory (RDGT).

**Algorithm RDGT**

1. Encode numerically the column and row players’ choices. These will be the inputs.
2. Encode numerically the payoffs (utilities) in the payoff table (matrix). These will be the outputs.
3. Represent the encoded inputs and outputs from steps (1) and (2) as a map (Look-Up-Table (LUT)).
4. If the encoded map is reversible, Then goto 6.
5. Else, apply the algorithm RevBF on the payoffs.
6. End.

**Example 2.** The extensive form, for the row and column players in the game in Figure (2a), is shown in Figure (2b).

![Figure 2. Irreversible game: (a) normal and (b) extensive forms.](image)

One can note that the extensive form in Figure (2b) is irreversible, i.e., an observer cannot reconstruct the input states from the output states, if the game extensive form is given, since the payoff cell (1, 3) appears twice. Following the algorithm RDGT, the following steps are performed:

1. Encode the players’ inputs (decisions; choices). This can be done as follows:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C1</th>
<th>C2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

2. Encode the outputs (payoffs) in the payoff matrix. This can be done as follows:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C1</th>
<th>C2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>1</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The map is clearly irreversible.

3. Obtain the map for steps 1 and 2:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>u_A</th>
<th>u_B</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The map is clearly irreversible.

4. Apply the RevBF algorithm:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>u_A</th>
<th>u_B</th>
<th>u_C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
<td>0</td>
<td>1</td>
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<td>1</td>
<td>0</td>
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</tbody>
</table>
The decision path is now reversible. The modified payoff matrix that corresponds to the reversible map in step (4) is now as follows:

<p>| | |</p>
<table>
<thead>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>0</td>
<td>000 001</td>
</tr>
<tr>
<td>1</td>
<td>010 100</td>
</tr>
</tbody>
</table>

and the reversible extensive form of the above payoff table is as follows:


One can observe that the encodings in steps (1) and (2) are not unique, i.e., other encoding schemes can be obtained as well. Note also that the new modified payoff matrix still reserves the dynamics of the original game, i.e., the dynamics do not change and the game’s solution does not change.

Since reversibility is a required property for Quantum Computing (QC) (Al-Rabadi, 2004; Bennett, 1973; Landauer, 1961; Nielsen and Chuang, 2000), the following Section introduces the use of MVQC to perform GT dynamics’ computations.

4. MULTIPLE-VALUED (m-VALUED) QUANTUM COMPUTING OF GAME THEORY

In addition to Prisoner’s Dilemma (PD) classical solutions using (Luce and Raiffa, 1989; Zagare, 1984): 1. iterated games, 2. non-myopic rationality and 3. metagames, recent development showed that the PD game can also be solved in the quantum domain by means of using Quantum Computing (QC) methods (Eisert, Wilkens and Lewenstein, 1999).

In this Section, the modeling of games’ dynamics using QC methods is implemented. In doing so, the following convention is used: 1. game dynamics is represented as linear unitary transformations; 2. players’ decisions are indices; 3. current payoffs (utilities) are inputs represented as input qudits; and 4. next payoffs (utilities) are outputs represented as output qudits.

We describe the notion of associating a quantum state to a point in the utility (payoff) space of a game using a 2-player m-utility game (i.e., two types of utilities each can potentially take up to m distinct values) as an example (extension to an n-player games is straightforward.) Assume each player’s utility can take a finite (discrete) set of values.

. Form a 2-dimensional grid of all possible combinations of utility value pairs (2-tuples).
. Assign each of the grid points (i.e., each 2-tuple) to be a quantum basis state (in the quantum state space).
. If each of the two utilities (u \(_i\) and u \(_j\)) can take m values, then there will be \(m^2\) quantum basis states, each with dimension \(m^2\) (to yield an orthonormal basis set).

Let:

\[
\vec{u}_{ij} = \begin{bmatrix} u_{i} \\ u_{j} \end{bmatrix}, \quad i, j = 0, 1, 2, ..., m - 1
\]

represent the \(m^2\) points in the 2-dimensional utility space, where the \((i, j)\) are position indices for the vector \(u_{ij}\) and the components of \(u_{ij}\) are utility values at the corresponding positions \((i, j)\).

Then define:

\[
\vec{\vec{u}}^{2: m} = \begin{bmatrix} \vec{u}_{00}^T \\ \vec{u}_{01}^T \\ \vdots \\ \vec{u}_{m-1,m-1}^T \end{bmatrix}
\]

where the superscript refers to 2 dimensions (2 players; 2 utilities), with \(m\) (discrete) values in each dimension. To reference a subset of all these possibilities, an appropriate subscript may be provided.

For example, by letting each utility take three values from the set \{a, b, c\} where a, b, c are any discrete real values (i.e., \(m = 3\)) then one would have nine grid points \{(00, 01, 02, 10, 11, 12, 20, 21, 22),\}, and Equation (14) becomes:

\[
\vec{\vec{u}}^{2: 3} = \begin{bmatrix} \vec{u}_{00}^T \\ \vec{u}_{01}^T \\ \vec{u}_{02}^T \\ \vec{u}_{10}^T \\ \vec{u}_{11}^T \\ \vec{u}_{12}^T \\ \vec{u}_{20}^T \\ \vec{u}_{21}^T \\ \vec{u}_{22}^T \end{bmatrix} = \begin{bmatrix} aa \\ ab \\ ac \\ ba \\ bb \\ bc \\ ca \\ cb \\ cc \end{bmatrix}
\]

Based on Equation (10), the following nine ternary orthonormal computational basis states in the Multiple-
Valued (MV) quantum space \( Q \) are obtained:

\[
\begin{align*}
|00\rangle &= |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
|01\rangle &= |0\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\
|02\rangle &= |0\rangle \otimes |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
|03\rangle &= |0\rangle \otimes |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
|10\rangle &= |1\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
|11\rangle &= |1\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\
|12\rangle &= |1\rangle \otimes |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
|13\rangle &= |1\rangle \otimes |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
|20\rangle &= |2\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
|21\rangle &= |2\rangle \otimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
|22\rangle &= |2\rangle \otimes |2\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
|23\rangle &= |2\rangle \otimes |3\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]

Note that this has resulted in an orthonormal basis set. One can perform MVQC by making the following assignments between the utility space \( U \) and the quantum space \( Q \): \( u^{2,3} \) = \( \begin{bmatrix} aa \\ ab \\ ac \\ ba \\ bc \\ ca \\ cb \\ ce \end{bmatrix} \) \( \Rightarrow \) \( \begin{bmatrix} 00 \\ 01 \\ 02 \\ 10 \\ 11 \\ 12 \\ 20 \\ 21 \\ 22 \end{bmatrix} \). \( |\psi(\psi^2)\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle + \gamma_1|2\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle + \gamma_2|2\rangle) \)

\[
= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_1\gamma_2|02\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle + \beta_1\gamma_2|12\rangle + \gamma_1\alpha_2|20\rangle + \gamma_1\beta_2|21\rangle + \gamma_1\gamma_2|22\rangle
\]

(16)

Note that each component of the tensor product is associated with a product of two probabilities.

The coefficients (probabilities) of the quantum basis functions (cf. Equation (4)) are the system parameters, obtained by solving the wave equation with the specified potential function \( V \) applied. We note that different \( V \)'s will (normally) result in different solutions (i.e., different probabilities) for each of the quantum basis states. Upon measurement of an observable variable in a physical quantum implementation, by definition, the highest probability state is the most likely one to occur (cf. Figure (5b)).

A quantum operator is a linear transformation, where the matrix of such transformation must be unitary (Al-Rabadi, 2004; Nielsen and Chuang, 2000) since matrix unitarity leads to computational reversibility. Each transformation in the quantum domain corresponds to specific type of logic gate (primitive) (Al-Rabadi, 2004; Nielsen and Chuang, 2000). Figure (3) shows examples of quantum gates and their quantum representations using linear unitary transformations.

One can use the quantum operators for modeling the maximin dynamics of a game. The following example shows how to model the dynamics of a 2-player game using MVQC.

**Example 3.** For the chicken game in Figure (1a), using the following encoding \( \{W = 0, T = 1, S = 2, B = 3\} \), the following is the 2-player interval-utility payoff matrix.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Figure 3. Binary quantum gates: (a) (2, 2) Feynman gate which uses XOR, (b) (3, 3) Toffoli gate which uses AND and XOR, and (c) (2, 2) Swap gate which are two permuted wires.
The payoffs in the above table are four valued, i.e., take values from the set {0, 1, 2, 3}. By using, in 4-valued quantum logic, the following qudits:

\[
\begin{align*}
&|00\rangle = |0\rangle |0\rangle, \quad |01\rangle = |0\rangle |1\rangle, \\
&|10\rangle = |1\rangle |0\rangle, \quad |11\rangle = |1\rangle |1\rangle,
\end{align*}
\]

then the combined utilities in the 4-valued quantum space take the following form:

\[
0000133131132222
\]

by solving the upper equations, one obtains the following set of unitary operators:

\[
\begin{align*}
\alpha = \frac{\delta - \gamma}{\alpha - \beta - \gamma + \delta} \\
\beta = \frac{\beta - \delta}{\beta + \gamma - \alpha - \delta} \\
\gamma = \frac{\alpha - \beta}{\alpha - \beta - \gamma + \delta} \\
\delta = \frac{\gamma - \alpha}{\beta + \gamma - \alpha - \delta}
\end{align*}
\]

The implementation of the above quantum model using quantum circuits (Al-Rabadi, 2004; Nielsen and Chuang, 2000) requires the serial interconnect of the quantum logic primitives \(\xi_1\) and \(\xi_5\), and the serial interconnect of the quantum logic primitives \(\xi_3\) and \(\xi_4\), as the serial interconnect in QC is implemented formally using regular matrix multiplication.

One notes that QC is used in Example (3) to model a pure strategy game. The next example shows the use of QC to model a mixed-strategy game and shows its quantum representations, using Quantum Decision Trees (QDTs) (Al-Rabadi, 2004).

**Example 4.** Let us consider the case of a simple 2-player mixed strategy ZSG as in Figure (4), where \(p\) is the probability of player A to choose decision \(A_1\), \((1-p)\) is the probability of player A to choose decision \(A_2\), \(q\) is the probability of player B to choose decision \(B_1\), and \((1-q)\) is the probability of player B to choose decision \(B_2\).

Using the probability definition in Equation (6), one can use the following QC assignments: \(p = |\alpha_1|^2,\ 1-p = |\beta_1|^2,\ p + (1-p) = |\alpha_1|^2 + |\beta_1|^2 = 1,\ q = |\alpha_2|^2,\ 1-q = |\beta_2|^2,\) and \(q + (1-q) = |\alpha_2|^2 + |\beta_2|^2 = 1.\) By encoding \(A_1 = "0",\ A_2 = "1",\ B_1 = "0",\) and \(B_2 = "1",\) then the modeling of each player’s decision in the game can be performed using qubit representation as follows:

\[
\begin{align*}
|\psi_A\rangle &= \alpha_1 |0\rangle + \beta_1 |1\rangle \\
|\psi_B\rangle &= \alpha_2 |0\rangle + \beta_2 |1\rangle
\end{align*}
\]

In terms of QC representation, solving the above game leads to the following equations:

\[
\begin{align*}
|\alpha_1|^2 + |\beta_1|^2 &= |\alpha_2|^2 + |\beta_2|^2 = 1, \\
\alpha_1 |\alpha_1|^2 + \beta_1 |\beta_1|^2 &= -\gamma |\alpha_2|^2 + \beta_2 |\beta_2|^2
\end{align*}
\]

Then the probability amplitudes \(|\alpha_1|, |\alpha_2|, |\beta_1|,\) and \(|\beta_2|\) of having decisions \(A_1, A_2, B_1,\) and \(B_2\) are, respectively:

\[
\begin{align*}
|\alpha_1|^2 &= \frac{\delta - \gamma}{\alpha - \beta - \gamma + \delta} \Rightarrow |\alpha_1| = \sqrt{\frac{\delta - \gamma}{\alpha - \beta - \gamma + \delta}} \\
|\alpha_2|^2 &= \frac{\beta - \delta}{\beta + \gamma - \alpha - \delta} \Rightarrow |\alpha_2| = \sqrt{\frac{\beta - \delta}{\beta + \gamma - \alpha - \delta}} \\
|\beta_1|^2 &= \frac{\alpha - \beta}{\alpha - \beta - \gamma + \delta} \Rightarrow |\beta_1| = \sqrt{\frac{\alpha - \beta}{\alpha - \beta - \gamma + \delta}} \\
|\beta_2|^2 &= \frac{\gamma - \alpha}{\beta + \gamma - \alpha - \delta} \Rightarrow |\beta_2| = \sqrt{\frac{\gamma - \alpha}{\beta + \gamma - \alpha - \delta}}
\end{align*}
\]

and the value of the game is:
\[ u^* = \frac{\alpha \delta - \beta \gamma}{\alpha - \beta - \gamma + \delta} \]  

(23)

As noted earlier, the quantum analog for the game dynamics are the dynamics described by the SE. Each cell of a game is represented as a single point in the utility quantum grid space (cf. Figure (5b)). An approach to implement a quantum game suggested here, is as follows (cf. Figure (5)): 1. specify maximin dynamics of game \( i \); 2. construct a separate wavefunction \( \psi_i \) in the MV quantum space for each cell such that its highest probability is at the joint maximin utility vector, and relatively low at all other utility vectors (cf. Figure (5b)); and 3. substitute this \( \psi_i \) into the TISE and solve for \( V_i \). Solutions of potential functions \( V_k \) for a collection of games \( G_k \) can be tabulated as a Look-Up-Table (LUT) where for each game (input) the output is the corresponding \( V \) to solve for the corresponding joint maximin dynamics (cf. Figure (5c)).

The quantum formulation in Figure (5) can be extended to a more general case: in a higher dimensional quantum space \( Q \), one would like to design for a potential vector \( V \) that solves for maximin dynamics of several games, i.e., potential with several maxima with each maximum corresponds to a specific joint maximin game dynamics, rather than several independent potential fields as in Figure (5c).

In general, for an \( n \)-player game, with each player has \( N \)-decisions, the quantum representation for each player will be in the form of Equation (12) for \( k = 0, \ldots, N-1 \).

For a game of two players, where each player has two decisions, each player decision affects the other. One can model all mutual (joint) decisions using QC, i.e., the superposition between the quantum states in Equations (17) and (18) (i.e., \( |\psi_{AB}\rangle \) ) can be implemented using Equation (7).

For the mixed-strategy game in Example (4), one can use the Quantum Decision Tree (QDT) (Al-Rabadi, 2004) as a quantum data representation as follows:

\[ |\psi_A\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} |0\rangle + \begin{bmatrix} 0 \\ 1 \end{bmatrix} |1\rangle \]

(24)

\[ |\psi_B\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} |0\rangle + \begin{bmatrix} 0 \\ 1 \end{bmatrix} |1\rangle \]

(25)

Where \( |1\rangle \) is the Buffer quantum operator (Al-Rabadi, 2004; Nielsen and Chuang, 2000). Quantum decision tree representations, using the computational basis states \( \{0\}, \{1\}\) for Equations (24), (25), and (7), are shown in Figure (6), where \( |\alpha| = \sqrt{p} \), \( |\beta| = \sqrt{1-p} \), \( |\alpha_2| = \sqrt{q} \), and \( |\beta_2| = \sqrt{1-q} \).

As an example, Figure (6c) shows the quantum decision path \( |AB\rangle = |01\rangle \) in a dashed dark line that leads to the highest probability \( \alpha_2 \beta_2 \) as a possible solution of the 2-player mixed strategy ZSG in Example (4).

The QDTs in Figure (6) use the quantum computational basis states to model a game’s dynamics. Other quantum basis states (Al-Rabadi, 2004; Nielsen and Chuang, 2000) such as the 1-qubit quantum systems’ orthonormal composite basis states:

\[ \left\{ \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right\} , \]

and the 2-qubit quantum systems’ Einstein – Podolsky –
Rosen (EPR) basis states:
\[
\begin{bmatrix}
|00\rangle + |11\rangle \\
|00\rangle - |11\rangle \\
|01\rangle + |10\rangle \\
|01\rangle - |10\rangle
\end{bmatrix}
\]
can be used (Al-Rabadi, 2004) for the quantum representation of games, where various tree representations will lead to different computational optimizations in terms of: 1. number of internal nodes used (i.e., memory used or spatial complexity) and 2. the speed of implementation operations using such representation (i.e., temporal complexity). For instance, by using the quantum Walsh-Hadamard operator
\[
\frac{1}{\sqrt{2}}\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\]
(Nielsen and Chuang, 2000), Equations (26) and (27) present the equivalence of Equations (24) and (25) in terms of the orthonormal composite basis states
\[
\begin{bmatrix}
|0\rangle + |1\rangle \\
|0\rangle - |1\rangle
\end{bmatrix}
\]
\[
\begin{bmatrix}
|+\rangle + |-\rangle \\
|+\rangle - |-\rangle
\end{bmatrix}
\]
Equations (26) and (27) in terms of the orthonormal composite basis states
\[
\begin{bmatrix}
|0\rangle + |1\rangle \\
|0\rangle - |1\rangle
\end{bmatrix}
\]
\[
\begin{bmatrix}
|+\rangle + |-\rangle \\
|+\rangle - |-\rangle
\end{bmatrix}
\]
where
\[
\begin{bmatrix}
|+\rangle = |0\rangle + |1\rangle \\
|\frac{1}{\sqrt{2}} | -\rangle
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
|0\rangle = |0\rangle + |1\rangle \\
|0\rangle - |1\rangle
\end{bmatrix}
\]
\[
\begin{bmatrix}
|+\rangle + |-\rangle \\
\frac{1}{\sqrt{2}} | +\rangle - | -\rangle
\end{bmatrix}
\]
Consequently, measuring \(|\Psi_A\rangle\) with respect to the new basis \(|+\rangle, |-\rangle\) will result in the state (basis) \(|+\rangle\) with probability \(\frac{|\alpha_1 + \beta_1|^2}{2}\) and the state (basis) \(|-\rangle\) with probability \(\frac{|\alpha_1 - \beta_1|^2}{2}\). Similarly, measuring \(|\Psi_B\rangle\) with respect to the new basis \(|+\rangle, |-\rangle\) will result in the state (basis) \(|+\rangle\) with probability \(\frac{|\alpha_2 + \beta_2|^2}{2}\) and the state (basis) \(|-\rangle\) with probability \(\frac{|\alpha_2 - \beta_2|^2}{2}\). Figure (7) shows the corresponding QDTs using Equations (26) and (27) for the game in Example (4).

As an example, Figure (7c) shows the quantum decision path \(|AB\rangle = |+\rangle |-\rangle\) in a dashed dark line that leads to the highest probability \(\lambda_1\mu_2\) as a possible solution of the 2-player mixed strategy ZSG in Example (4).

For the more general case, Figures (6) and (7) can be extended to several levels and several branches per level, where: 1. number of levels equals the number of players, and 2. number of branches per level is equal to the number of decisions per player. Such generalized QDTs are called Multiple-Valued QDTs (MvQDTs) (Al-Rabadi, 2004).

![Figure 7. Orthonormal composite basis states QDT representation for: (a) Equation (26), (b) Equation (27), and (c) Equation (7).](image-url)
\[ |\psi_{12...n}\rangle \neq \prod_{p=1}^{n} \left( \sum_{k=1}^{d} \alpha_k |D_k\rangle \right)_p \] (30)

If the state vectors of certain quantum systems (e.g., players in the GT context) were entangled with each other, then if one changes the state vector of one system, the corresponding state vector of the other system is also changed, instantaneously and independently, of the medium through which some communicating signal must travel where there are no forces involved. This exact phenomenon has a solid natural analogue that can be used for GT modeling purposes: when two photons are created, and their spin conserved, as it must, one photon has a spin of 1 and a spin of -1. By measuring one of the state vectors of the photon, the state vector collapses into a knowable state. Instantaneously and automatically, the state vector of the other photon collapses into the other knowable state. When one photon’s spin is measured and found to be 1, the other photon’s spin of -1 immediately becomes known too.

5. CONCLUSIONS AND FUTURE WORK

This paper introduces the new concept of Reversible Game Theory (RGT) and its \(m\)-Valued Quantum Computation (MVQC).

Reversibility in GT can be important in situations where 1. an outside observer needs to know reversely the decision paths that lead to certain payoffs on the game’s extensive form, and 2. modeling the game’s maximin dynamics using low-power consuming circuits since reversibility is a main requirement for low-power circuit design of future technologies such as in Quantum Computing (QC).

Quantum representations in GT of 1. the orthonormal computational basis states Quantum Decision Trees (QDTs) and 2. orthonormal composite basis states QDTs are also presented as quantum representations for the modeling and manipulations of games’ dynamics.

Future work will include more investigations into the application of reversibility and QC to items such as: 1. games’ separability (decomposition); 2. \(N\)-player games (e.g., \(N\)-player PD (tragedy of the commons)); 3. multi-level utilities; 4. evolutionary GT; 5. altruism in GT; 6. situations of bargaining, negotiation, and conflict resolution; 7. NZSG and 8. coalition theory. Also, more investigation for the use of redundant quantum systems (e.g., atoms) to account for the error correction of QC within the context of GT will be conducted.

APPENDIX

A physical system comprising trapped ions under multiple laser excitations can be used to reliably implement MVQC (Muthukrishnan and Stroud, 2000). A physical system in which an atom (particle) is exposed to a specific potential field (function) \(V(x)\) can also be used to implement MVQC (two-valued being a special case) (Al-Rabadi, 2004; Nielsen and Chuang, 2000). In such an implementation, the (resulting) distinct energy states are used as the orthonormal basis states. The latter is illustrated in Example (5) below.

Example 5. We assume the following constraints:

1. spring potential \(V(x) = (1/2) kx^2\), where: \(m\) is a particle, \(k = m\alpha^2\) is spring constant and \(\omega\) is angular frequency (= \(2\pi\cdot\text{frequency}\)), and (2) boundary conditions. Also, assuming the solution of the TISE in Equation (3) for these constraints is of the following form (i.e., the Gaussian function):

\[ \psi(x) = Ce^{-\frac{x^2}{\alpha^2}} \]

where: \(\alpha = \frac{m\omega}{\hbar / 2\pi}\). The general solution for the wave function \(|\psi\rangle\) (for a spring potential) is:

\[ C = \left[ \frac{\alpha}{\pi} \right]^{1/4} \frac{1}{\sqrt{2^n!}} H_n(\sqrt{\alpha}x) \]

where \(H_n(x)\) are the Hermite polynomials. This solution leads to the sequence of evenly spaced energy levels (eigenvalues) \(E_n\) characterized by a quantum number \(n\) as follows:

\[ E_n = (n + \frac{1}{2})(\hbar / 2\pi)\omega \]

The distribution of the energy states (eigenvalues) and their associated probabilities are shown in Figure (8).
Figure 8. Harmonic Oscillator (HO) potential and wavefunctions: (a) wavefunctions for various energy levels (subscripts), (b) spring potential $V(x)$ and the associated energy levels $E_n$, and (c) probabilities for measuring particle $m$ in each energy state ($E_n$).

REFERENCES


