Dirasat, Pure Sciences, Volume **39**, No. **1**, 2013

The Effects of Treatment and Immigrants on the Dynamics of SIS Epidemic Model

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ABSTRACT

In this paper, a mathematical model representing *SIS* epidemic model involving a limited resource of treatment and immigrants has been proposed and analyzed. It is assumed that fraction of the immigrants comes infected. The boundedness of the system is discussed. The local stability analysis of all possible equilibrium points is investigated. The global stability of the system is studied analytically as well as numerically. Finally, it is observed that, the existence of treatment and the immigrants have a stabilizing effect on the dynamical behavior of the proposed model.

Keywords: SIS- epidemic Model, Treatment, Immigrants, Local stability, Global Stability.

INTRODUCTION

Over the past one hundred years, mathematics has been used to understand and predict the spread of diseases. Almost all mathematical models of disease start from the same basic premise: that the population can be sub divided into a set of distinct classes, dependent upon their experience with respect to the disease. One line of investigation classifies individuals as an *SIS* model. The *SIS* model is one of the most basic and most important models in describing the spread of many diseases. A detailed history of mathematical epidemiology and basics of *SIS* epidemic model can be found in many books and paper. Early Kermack and Mckendrick (1927) proposed a simple *SIS* model with infective immigrants. Gao and Hethcote (1995) considered the *SIS* model with a standard disease incidence and density-dependent demographics. Li and Ma (2002) studied the *SIS* model with vaccination and temporary immunity. Zhou and Liu (2003) considered an *SIS* model with pule vaccination. Thus a full understanding of the *SIS* model is essential regardless of how well any particular disease can be forced into its frame-work.

The treatment is an important method to decrease the spread of disease such as measles, tuberculosis and flu (Feng and thieme, 1995) and (Hyman and Li, 1998). Therefore, it attached many authors` attention and a number of papers have been published. Li et al. (2009) proposed the *SIS* model with a limited resource for treatment. It is assumed that treatment rate is proportional to the number of infected individuals below the capacity and a constant when the number of infected individuals is greater than the capacity. It is found that a backward bifurcation occurs if the capacity is small.

In this paper, the *SIS* model with a limited resource for treatment of Li et al. (2009) is reformulated so that it involves immigrant. It is assumed that fraction of the immigrants comes infected. The effect of immigrants on the dynamical behavior of the new resulting model is considered analytically as will as numerically. The stability (local as well as global) analysis of all possible equilibrium points are established for this model.

Mathematical Model

In this section, an epidemic population with a limited resource for treatment is proposed for study. The population is divided into two classes: the susceptible individuals $S(t)$ at time t, and the infected individuals $I(t)$ at time t. It is assumed that a constant flow, say A, of new members arrives into the population in unit time with the fraction p ($0 \le p \le 1$) of A arriving infected. Accordingly the dynamics of *SIS* epidemic model with a limited resource for treatment and constant rate of immigrants, which represented in the block diagram

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given by Fig. (1), can be represented by the following system of non linear ordinary differential equations:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda + (1-p)A - \mu S - \beta SI + \gamma I + T(I) \\ \frac{dI}{dt} &= pA + \beta SI - (\mu + \alpha + \gamma)I - T(I) \end{aligned} \quad (1)$$

Note that the parameters Λ, A, μ, γ, α and β of system (1) are assumed to be positive and can be described as follows: Λ is the recruitment rate of the susceptible population; A is the constant flow rate of immigrants; μ is the natural death rate in each class; γ is the nature recovery rate of infected individuals; α is the disease related death rate; β is the infection coefficient; finally $T(t)$ is the treatment function which given by (Wang, 2006):

$$T(I) = \begin{cases} rI & if \quad 0 < I \le I_0 \\ K & if \quad I > I_0 \end{cases} \quad (2)$$

here $K = rI_0$ this means that the treatment rate is proportional to the number of the infected individuals when the capacity of treatment is not reached, and otherwise takes the maximal capacity.

Obviously, due to the biological meaning of the components $S(t)$ and $I(t)$ we focus on the model in the domain $R_+^2 = \{(S,I) \in R^2 : S \ge 0, I \ge 0\}$ which is positively invariant for system (1).

Theorem (1): All the solutions of system (1) which initiate in R_+^2 are uniformly bounded.

Proof: Let $(S(t), I(t))$ be any solution of the system (1) with non negative initial condition ($S(0), I(0)$). Assume that $N(t) = S(t) + I(t)$, then

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} = \Lambda + A - \mu S - \mu I - \alpha I \Rightarrow$$

$$\frac{dN}{dt} \le \Lambda + A - \mu(S + I)$$

So, $\frac{dN}{dt} + \mu N \le \Lambda + A = B$

Now, due to the theory of differential inequalities (see Hale, 1969), we obtain

$$N(t) \le \frac{B}{\mu}(1 - e^{-\mu t}) + N(0)e^{-\mu t}$$

Therefore $N(t) \le \frac{B}{\mu}$, as $t \to \infty$ Hence all the solutions of system (1) that initiate in R_+^2 are confined in the region $\Omega = \left\{ (S,I) \in R_+^2 : N = S + I \le \frac{B}{\mu} \right\}$, thus these solutions are uniformly bounded and then the proof is complete.■

In order to understand the effect of treatment and immigrant on SIS epidemic model we will starts to study the dynamical behavior of SIS epidemic model without treatment and immigrants in the following section.

Stability analysis of SIS epidemic model without treatment and immigrant

System (1) without treatment and immigrant can be written in the following form:

$$\begin{aligned} \frac{dS}{dt} &= \Lambda - \beta SI - \mu S + \gamma I \\ \frac{dI}{dt} &= \beta SI - (\mu + \alpha + \gamma)I \end{aligned} \quad (3)$$

Now, the existence and stability analysis of all possible equilibrium points of system (3) are discussed.

For any set of parameters, system (3) has always disease free equilibrium point $E = (\hat{S}, 0) = (\frac{\Lambda}{\mu}, 0)$. However, an endemic equilibrium point of system (3) satisfies

$$\begin{aligned} \Lambda - \beta SI - \mu S + \gamma I &= 0 \\ \beta SI - (\mu + \alpha + \gamma)I &= 0 \end{aligned} \quad (4)$$

Now, straight forward computation gives that system (4) has a unique positive solution given by:

$$S_* = \frac{(\mu + \alpha + \gamma)}{\beta};\ I_* = \frac{\Lambda\beta - \mu(\mu + \alpha + \gamma)}{\beta(\mu + \alpha)} \quad (5)$$

Clearly, I_* is positive under the following condition:

$$\Lambda\beta \succ \mu(\mu + \alpha + \gamma) \quad (6)$$

Therefore, $E_* = (S_*, I_*)$ is an endemic equilibrium point of system (3). Now, the local dynamical behavior of system (3) around each of these equilibrium points is discussed, and the following results are obtained.

The Jacobian matrix of system (3) at the disease free equilibrium point $E = (\hat{S}, 0)$ is given by:

$$J(E) = \begin{bmatrix} -\mu & \frac{-\beta\Lambda}{\mu} + \gamma \\ 0 & \frac{\beta\Lambda}{\mu} - (\mu + \alpha + \gamma) \end{bmatrix}$$

Clearly the eigenvalues of $J(E)$ are $\lambda_1 = -\mu \prec 0$, $\lambda_2 = \frac{\beta\Lambda}{\mu} - (\mu + \alpha + \gamma)$. Therefore, E is locally asymptotically stable provided that the following condition holds:

$$\beta\Lambda \prec \mu(\mu + \alpha + \gamma) \quad (7)$$

While, the Jacobian matrix of system (3) at the endemic equilibrium point $E_* = (S_*, I_*)$ is given by:

$$J(E_*)=\begin{bmatrix}-\beta I_*-\mu & -(\mu+\alpha)\\ \beta I_* & 0\end{bmatrix}$$

Therefore we have that

$T=-\beta I_*-\mu \prec 0$

$D=(\mu+\alpha)\beta I_* \succ 0$

So, according to the trace-determinate $(T-D)$ stability criterion, E_* is locally asymptotically stable in the $Int.R_+^2$.

In the following the global dynamics of system (3) is carried out as shown in the following theorems.

Theorem (2): Assume that the disease free equilibrium point $E=(\hat{S},0)$ of system (3) is locally asymptotically stable in R_+^2 and let that

$$\beta\hat{S}\prec(\mu+\alpha) \quad (8)$$

Then E is globally asymptotically stable in R_+^2.

Proof: Consider the following positive definite function

$$U(S,I)=\left(S-\hat{S}-\hat{S}\ln\frac{S}{\hat{S}}\right)+I$$

Clearly, $U:R_+^2\to R$ is continuously differentiable function such that $U(\hat{S},0)=0$ and $U(S,I)\succ 0,\ \forall(S,I)\in R_+^2$ with $(S,I)\neq(\hat{S},0)$. Moreover, since

$$\frac{dU}{dt}=\left(\frac{S-\hat{S}}{S}\right)\frac{dS}{dt}+\frac{dI}{dt}$$

Then by substituting the values of $\frac{dS}{dt},\frac{dI}{dt}$ in the above equation and simplifying the resulting terms we get:

$$\frac{dU}{dt}\le\frac{-\mu}{S}\left(S-\hat{S}\right)^2+\left(\beta\hat{S}-(\alpha+\mu)\right)I$$

Hence, $\frac{dU}{dt}\prec 0$ under the sufficient condition (8), and then U is a Lyapunov function. Therefore E is globally asymptotically stable in R_+^2. ■

It is well known that, the existence of limit cycles for any bounded two-dimensional system defined on $D\subseteq R^2$ plays a crucial role on the structure of dynamical behavior of that system. In fact, if the system has no limit cycle in interior of D surrounding its unique equilibrium point, then according to the Poincare-Bendixson Theorem this equilibrium point is globally asymptotically stable in the interior of D. For this reason Dulac function is adopted to obtain condition for the non existence of a limit cycle in the $Int.R_+^2$ of system (3).

Theorem (3): Assume that system (3) has a unique endemic equilibrium point $E_*=(S_*,I_*)$ in the $Int.R_+^2$, then it is globally asymptotically stable in the $IntR_+^2$.

Proof: Consider a Dulac function $D=\frac{1}{SI}$ and assume that

$$g_1=\frac{dS}{dt}=\Lambda-\beta SI-\mu S+\gamma I$$

$$g_2=\frac{dI}{dt}=\beta SI-(\mu+\alpha+\gamma)I$$

Clearly $D(S,I)\succ 0$ for all $S,I\in IntR_+^2$ and it is C^1 function in the $IntR_+^2$. Now, since

$$\Delta(S,I)=\frac{\partial Dg_1}{dS}+\frac{\partial Dg_2}{dI}=\frac{-\Lambda}{S^2I}-\frac{\gamma}{S^2}\prec 0$$

Note that $\Delta(S,I)$ dose not change sign and is not identically zero in the $Int.R_+^2$. Then according to Bendixon-Dulac criterion, there is no periodic solution in the $Int.R_+^2$. Now since all the solutions of the system (3) are bounded and E_* is a unique positive equilibrium point in the $Int.R_+^2$, hence by using the Poincare-Bendixon theorem E_* is globally asymptotically stable.

■

Equilibrium points and local stability analysis of system (1)

In this section the existence of all possible equilibrium points of system (1) and their local stability analysis is discussed. For any set of parameters, system (1) has always disease free equilibrium point $E=\left(\frac{\Lambda+A}{\mu},0\right)$. However an endemic equilibrium point of system (1) satisfies that:

$$\begin{aligned}&\Lambda+(1-P)A-\mu S-\beta SI+\gamma I+T(I)=0\\&PA+\beta SI-(\mu+\alpha+\gamma)I-T(I)=0\end{aligned} \quad (9)$$

Clearly, when $0\prec I\le I_0$, system (9) becomes :

$$\begin{aligned}&\Lambda+(1-p)A-\mu S-\beta SI+\gamma I+rI=0\\&pA+\beta SI-(\mu+\alpha+\gamma+r)I=0\end{aligned} \quad (10a)$$

While, for $I\succ I_0$, system (9) becomes :

$$\begin{aligned}&\Lambda+(1-p)A-\mu S-\beta SI+\gamma I+K=0\\&pA+\beta SI-(\mu+\alpha+\gamma)I-K=0\end{aligned} \quad (10b)$$

Now, straight forward computation gives that, system (10a) has a unique positive solution given by:

$$\begin{aligned}S^*&=\frac{\Lambda+(1-p)A+(\gamma+r)I^*}{\mu+\beta I^*}\\ I^*&=\frac{-a_2-\sqrt{a_2^2-4a_1a_3}}{2a_1}\end{aligned} \quad (11)$$

where

$a_1=-\beta(\mu+\alpha)$, $a_2=\beta(\Lambda+A)-\mu(\mu+\alpha+\gamma+r)$ and $a_3=pA\mu$. Therefore $E^*=(S^*,I^*)$ is an endemic equilibrium point of system (1) when $0 \prec I \leq I_0$. On the other hand when $I \succ I_0$, system (10b) has the solution

$$\bar{S} = \frac{\Lambda + (1-p)A + \gamma\bar{I} + K}{\mu + \beta\bar{I}} \qquad (12a)$$

While, $\bar{I}$ is the positive root of the following second order equation

$$aI^2 + bI + c = 0 \qquad (12b)$$

where $a = -\beta(\mu+\alpha)$, $b = \beta(\Lambda + A) - \mu(\mu+\alpha+\gamma)$ and $c = \mu(pA - K)$. Then we have the following three cases:

Case 1: If the following condition holds:

$$c > 0 \Leftrightarrow pA > K \qquad (13a)$$

then equation (12b) has a unique positive root is given by :

$$\bar{I} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad (13b)$$

Accordingly, system (1) has a unique endemic equilibrium point $\bar{E} = (\bar{S}, \bar{I})$ when $I \succ I_0$ given by equations (12a) and (13b) provided that condition (13a) holds.

Case 2: If the following condition holds:

$$c < 0 \Leftrightarrow pA < K \qquad (13c)$$

then we have the following two sub cases :

Sub case 1: If the following condition holds:

$$b < 0 \Leftrightarrow \beta(\Lambda + A) < \mu(\mu+\alpha+\gamma) \qquad (13d)$$

then equation (12b) has no positive root and hence system (1) has no endemic equilibrium point .

Sub case 2: If the following condition holds:

$$b > 0 \Leftrightarrow \beta(\Lambda + A) > \mu(\mu+\alpha+\gamma) \qquad (13e)$$

and

$$\Delta = b^2 - 4ac > 0 \qquad (13f)$$

Then equation (6b) has two positive roots given by:

$$\tilde{I}_1 = \frac{-b + \sqrt{\Delta}}{2a} \qquad (14a)$$

$$\tilde{I}_2 = \frac{-b - \sqrt{\Delta}}{2a} \qquad (14b)$$

Consequently, provided that conditions (13c), (13e) and (13f) hold system (1) has two endemic equilibrium points in the interior of R_+^2 (i.e $Int.R_+^2$), which are given by $\tilde{E}_1 = (\tilde{S}_1, \tilde{I}_1)$ and $\tilde{E}_2 = (\tilde{S}_2, \tilde{I}_2)$, where $\tilde{S}_i = \bar{S}(\tilde{I}_i);\ i = 1,2$.

Case 3: If the following condition holds:

$$c = 0 \Leftrightarrow pA = K \qquad (15a)$$

then equation (12b) has a unique positive root is given by :

$$\breve{I} = \frac{-b}{a} \qquad (15b)$$

provided that the condition (13e) holds. Then $\breve{E} = (\breve{S}, \breve{I})$ with $\breve{S} = \bar{S}(\breve{I})$ is an endemic equilibrium point of system (1), which is exists if and only if conditions (13e) and (15a) are satisfied.

Now the dynamical behavior of system (1) around each of these equilibrium points is studied and the following results are obtained.

The Jacobian matrix of system (1), in the case of $0 \prec I \leq I_O$, at the disease free equilibrium point $E = \left(\frac{\Lambda + A}{\mu}, 0\right)$ is given by:

$$J(E) = \begin{bmatrix} -\mu & \frac{-\beta(\Lambda + A)}{\mu} + (\gamma + r) \\ 0 & \frac{\beta(\Lambda + A)}{\mu} - (\mu+\alpha+\gamma+r) \end{bmatrix}$$

Clearly the eigenvalues of $J(E)$ are given by $\lambda_1 = -\mu < 0$ and $\lambda_2 = \frac{\beta(\Lambda+A)}{\mu} - (\mu+\alpha+\gamma+r)$, therefore E is locally asymptotically stable provided that:

$$\frac{\beta(\Lambda + A)}{\mu} \prec (\mu+\alpha+\gamma+r) \qquad (16a)$$

Similarly, in case of $I \succ I_0$, the eigenvalues of $J(E)$ are given by $\lambda_1 = -\mu < 0$ and $\lambda_2 = \frac{\beta(\Lambda+A)}{\mu} - (\mu+\alpha+\gamma)$. Therefore E is locally asymptotically stable provided that:

$$\frac{\beta(\Lambda + A)}{\mu} \prec (\mu+\alpha+\gamma) \qquad (16b)$$

Obviously, if condition (16b) holds then condition (16a) holds too and then the disease free equilibrium point of system (1) is locally asymptotically stable for the treatment function *T(I)* given by (2) if and only if condition (16b) holds.

The Jacobian matrix of system (1), in the case $0 \prec I \leq I_0$, at the endemic equilibrium point $E^* = (S^*, I^*)$ is computed as follows $J(E^*) = \left(a_{ij}\right)_{2\times 2}$, where

$$a_{11} = -\mu - \beta I^*,\ a_{12} = -\beta S^* + (\gamma + r)$$

$$a_{21} = \beta I^*,\ a_{22} = \beta S^* - (\mu+\alpha+\gamma+r)$$

Therefore we have that:

$$D(J(E^*)) = a_{11}a_{22} - a_{12}a_{21} \qquad (17a)$$

$$T(J(E^*)) = a_{11} + a_{22} \qquad (17b)$$

Now, according to the trace-determinate stability criterion, E^* is locally asymptotically stable if and only if $D(J(E^*)) > 0$ and $T(J(E^*)) < 0$.

Consequently, it is easy to verify that the endemic equilibrium point E^* of system (1) for the case

$0 \prec I \le I_0$ is locally asymptotically stable if and only if the following sufficient condition holds:

$$\frac{(\gamma + r)}{\beta} < S^* < \frac{(\mu + \alpha + \gamma + r)}{\beta} \qquad (17c)$$

Otherwise the equilibrium point $E^* = (S^*, I^*)$ is unstable point and the solution of system (1) will approach to periodic solution.

The Jacobian matrix of system (1) for the case $I \succ I_0$ at the endemic equilibrium point $\bar{E} = (\bar{S}, \bar{I})$ is given by $J(\bar{E}) = (b_{ij})_{2\times 2}$, where

$b_{11} = -\mu - \beta\bar{I}$, $b_{12} = -\beta\bar{S} + \gamma$, $b_{21} = \beta\bar{I}$, $b_{22} = \beta\bar{S} - (\mu + \alpha + \gamma)$

Accordingly we have that:

$$D.(J(\bar{E})) = b_{11}b_{22} - b_{21}b_{12} \qquad (18a)$$

$$T(J(\bar{E})) = b_{11} + b_{22} \qquad (18b)$$

hence the endemic equilibrium point $\bar{E} = (\bar{S}, \bar{I})$ of system (1) for $I \succ I_0$ is locally asymptotically stable if and only if the following sufficient condition is satisfied :

$$\frac{\gamma}{\beta} < \bar{S} < \frac{(\mu + \alpha + \gamma)}{\beta} \qquad (18c)$$

Otherwise it is unstable point and periodic dynamics exists. Similarly, it is easy to verify that, the endemic equilibrium points $\tilde{E}_1, \tilde{E}_2$ and $\breve{E}$ of system (1) for $I \succ I_0$ (whenever they exist) are locally asymptotically stable under condition (18c) with replacing $\bar{E}$ by $\tilde{E}_1, \tilde{E}_2$ and $\breve{E}$ in equations (18a) and (18b).

Global Stability Analysis

In this section the global dynamics of system (1) is carried out and the obtained results are shown in the following theorems.

Theorem (4): The disease free equilibrium point $E = \left(\frac{\Lambda + A}{\mu}, 0\right) = (S_0, 0)$ of system (1) is globally asymptotically stable in R_+^2 provided that

$$\beta S_0 \prec (\mu + \alpha) \qquad (19)$$

Proof: Assume that $0 \prec I \le I_0$ (similarly proof for $I \succ I_0$). Consider the following positive definite function about (S_0, 0).

$$W(S, I) = \left(S - S_0 - S_0 \ln \frac{S}{S_0}\right) + I$$

Clearly $W : R_+^2 \to R$ is continuously differentiable function such that $W(S_0, 0) = 0$ and $W(S, I) > 0$ otherwise. Moreover, since

$$\frac{dW}{dt} = \frac{(S - S_0)}{S}\frac{dS}{dt} + \frac{dI}{dt}$$

Then by substituting the values of $\frac{dS}{dt}, \frac{dI}{dt}$ in the above equation and simplifying the resulting terms we get

$$\frac{dW}{dt} \le \frac{-\mu}{S}(S - S_0)^2 - (\mu + \alpha - \beta S_0)I$$

Therefore, for all values of $(S, I) \in R_+^2$, condition (19) represents the sufficient condition for $\frac{dW}{dt} < 0$ and hence *W* is a Lyapunov function, therefore *E* is globally asymptotically stable in R_+^2 . ■

In the following theorem the global dynamics of system (1) in $Int.R_+^2$ is carried out by using Bendixon-Dulac criterion.

Theorem (5): Assume that system (1) has a unique endemic equilibrium point in the $Int.R_+^2$ which is locally stable, then it is globally asymptotically stable in the $Int.R_+^2$.

Proof: Assume that $0 \prec I \le I_0$ (similar proof for $I \succ I_0$) Consider a Dulac function $D = \frac{1}{SI}$ and assume that:

$$f_1 = \Lambda + (1-p)A - \mu S - \beta SI + (\gamma + r)I$$

$$f_2 = pA + \beta SI - (\mu + \alpha + \gamma + r)I$$

Clearly $D(S, I) \succ 0$ for all $(S, I) \in Int.R_+^2$, and its C^1 function in $Int.R_+^2$. Now, since:

$$\Delta(S, I) = \frac{\partial(Df_1)}{\partial S} + \frac{\partial(Df_2)}{\partial I} = \frac{-\Lambda}{S^2 I} - \frac{(1-p)A}{S^2 I} - \frac{(\gamma + r)}{S^2} - \frac{pA}{SI^2}$$

Note that $\Delta(S, I)$ dose not change sign and is not identically zero in the $Int.R_+^2$. Then according to Bendixson-Dulac criterion there is no periodic solution in the $Int.R_+^2$. Now, since all the solutions of the system (1) are bounded and E^* is a unique positive equilibrium point in the $IntR_+^2$, hence by using the Poincare-Bendixson Theorem E^* is globally asymptotically stable. ■

Numerical analysis

In this section the global dynamics of system (1) is studied numerically. The objectives of this study are confirming our analytical results and understand the effects of treatment and immigration on the dynamics of *SIS* epidemic system. Consequently, system (1) is solved numerically, for different sets of parameters and different sets of initial conditions.

It is observed that, for the following set of parameters, system (1) is solved for different sets of initial values and then the trajectories of system (1) as a function of time are drawn Fig. (2a)-(2b).

$$\Lambda = 400, A = 100, p = 0, \mu = 0.1, \ \beta = 0.0001, \alpha = 2, \gamma = 2, r = 1, I_0 = 75 \quad (20)$$

Obviously, Fig. (2) shows clearly the convergent of system (1) to the globally asymptotically stable disease free point $E = (5000, 0)$, which confirm our analytical results. However, for the following set of parameters:

$$\Lambda = 400, A = 100, p = 0.01, \mu = 0.1, \ \beta = 0.01, \alpha = 2, \gamma = 2, r = 1, I_0 = 75 \quad (21)$$

The phase plot of system (1) starting from different sets of initial data is drawn in Fig. (3).

Clearly, Fig. (3) shows the existence of a unique endemic equilibrium point of system (1) which is globally asymptotically stable.

Now, in order to discuss the effect of varying the infection rate β on the dynamical behavior of system (1), the system (1) is solved for different values of the infection rate $\beta = 0.002, 0.005, 0.02$ keeping other parameters as given in Eq. (21), and the solution of system (1) is drawn in Fig. (4a)-(4c).

According to the above figure, it is clear that, as the infection rate decreases the endemic equilibrium point losses it is stability and a Hopf bifurcation occurs as shown in Fig. (4a). However, as it increases the endemic equilibrium point still coexists and stable with increase in the value of infected individuals whereas the value of susceptible individuals decreases.

Now the effect of varying the rate of infected immigrant individuals p on the dynamics of system (1) is studied. So, system (1) is solved for the parameter values $p = 0.001, 0.5, 1$ keeping other parameters fixed as given in Eq. (21), and then the trajectories of system (1) are drawn as a function of time in Fig. (5a)-(5b).

According to this figure, as the rate of infected immigrant individuals increases the endemic equilibrium point of system (1) still coexists and stable but the number of susceptible individuals decreases whereas the number of infected individuals increases slightly.

The effect of varying the treatment rate on the dynamics of system (1) is studied and the trajectories of system (1) are drawn in Fig. (6a)-(6c) for the values $r = 3,\ 4,\ 5.24$ respectively.

Obviously, from these figures as the treatment rate increases the endemic equilibrium point of system (1) losses its stability and the solution approaches to limit cycle. In fact it is observed that a Hopf bifurcation accurse at $r = 5.24$.

Finally, it is observed that, increasing the value of the maximum value of infected individuals in the treatment function (the value of I_0) or the value of death rate due the disease (the value of α) causes increasing in S and decreasing in I but the system (1) still approaches to endemic equilibrium point.

Conclusions

In this paper, a mathematical model has been proposed and analyzed to study the effect of treatment and immigrants on the dynamical behavior of *SIS* epidemic model. The dynamical behavior of system (1) has been investigated locally as well as globally. It is observed (analytically) that when the endemic equilibrium point of the *SIS* model without treatment and immigrants exists, it is always globally asymptotically stable. On the other hand it is observed that (analytically as well as numerically) when the endemic equilibrium point of *SIS* model with treatment and immigrant exists then it is either globally asymptotically stable or unstable and hence the solution approaches to a periodic attractor surrounding it.

Now, we shall discuss the effects of changing the parameters on the dynamics of system (1) according to the numerical results in section (6):

1. For small value of infection rate β (for example $\beta = 0.005$) the trajectory of system (1) approaches to a periodic attractor. However, as β increases the trajectory of system (1) approaches to global asymptotically stable point (endemic point) in the $Int.R_+^2$.
2. The trajectory of system (1) always approach to global asymptotically stable point (endemic point) in the $Int.R_+^2$ for different values of p, I_0, and α.
3. For small value of treatment rate r the trajectory of system (1) is approach to a global asymptotically stable point (endemic point) in the $Int.R_+^2$, while it approaches to a periodic attractor for $r \succ 5.24$.

Clearly, the existence of treatment and immigrants, as shown in this paper, plays an important role in stabilizing the dynamics of *SIS* epidemic model for some values of parameters sets.

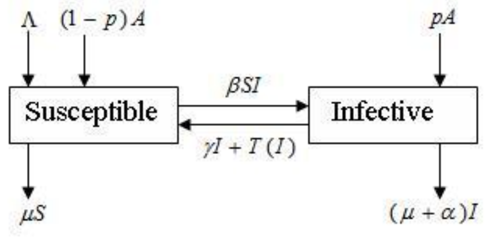

Fig. 1. Block diagram for the SIS epidemic model with a limited resource for treatment and constant rate of immigrants

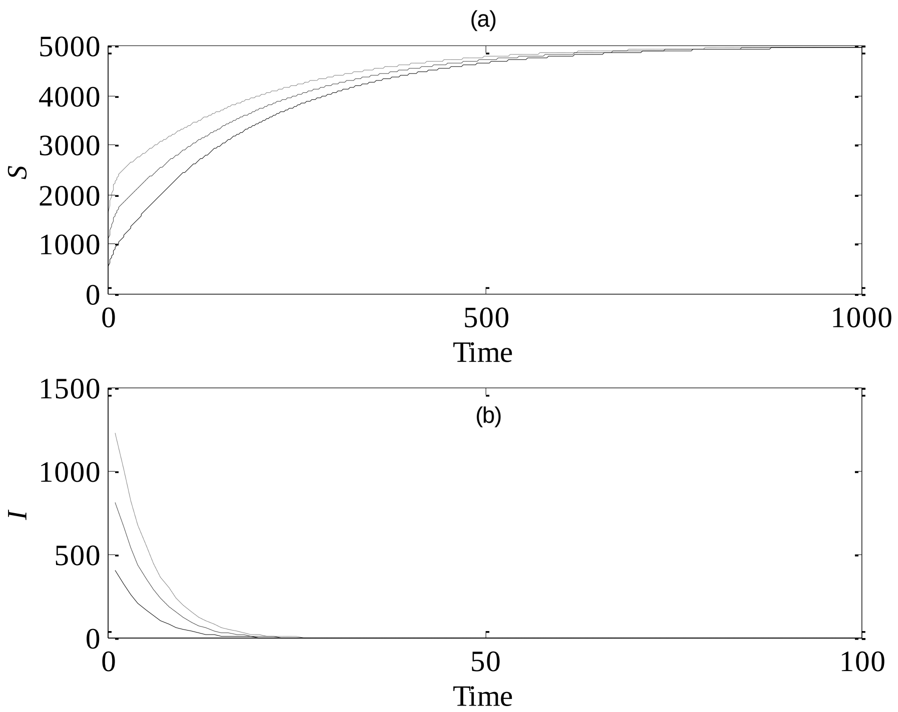

Fig. 2. Time series of the solutions of system (1); Solid line is the trajectory started at (500, 500); dash line is the trajectory started at (1000, 1000); dot line is the trajectory started at (1500, 1500). (a) Trajectory of S as a function of time. (b) Trajectory of I as a function of time

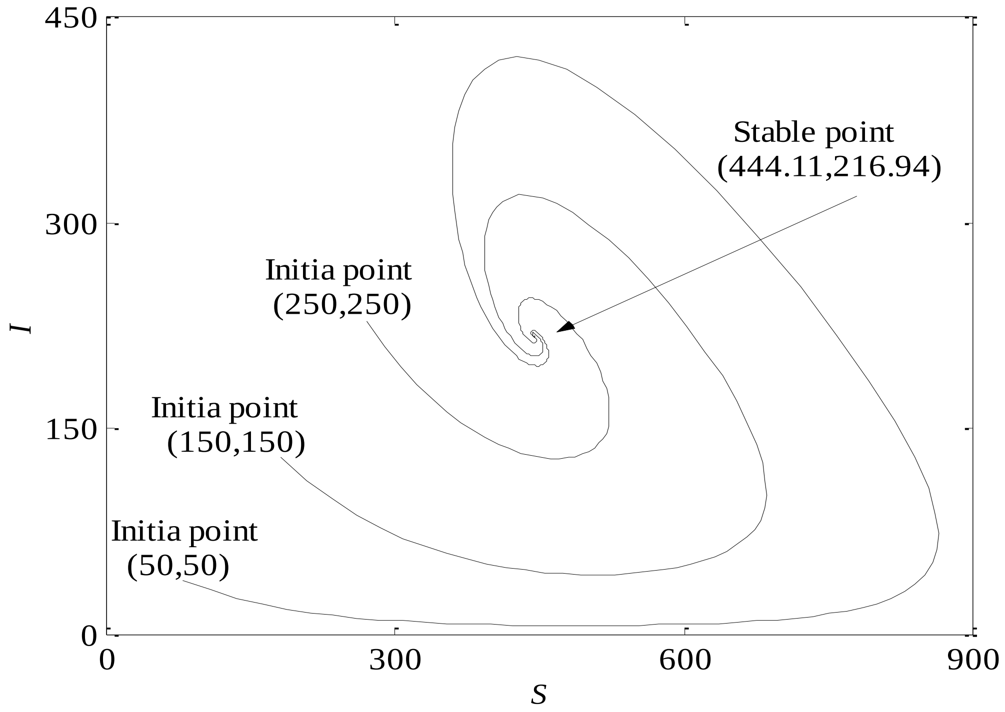

Fig. 3. Phase plot of system (1) starting from different initial points

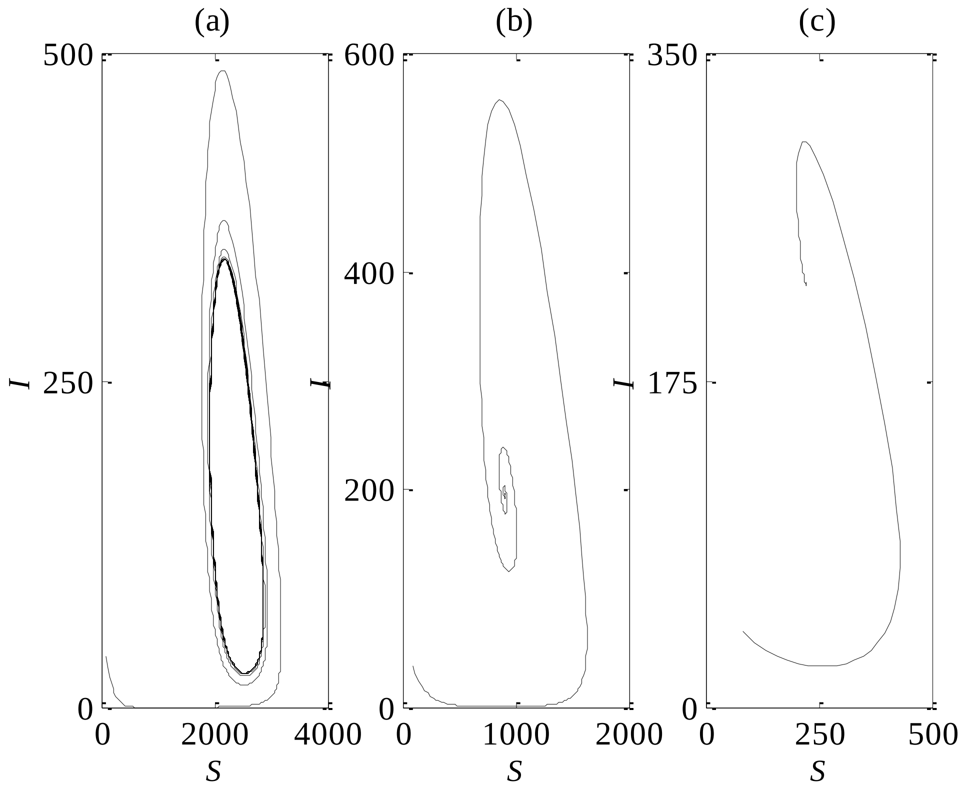

Fig. 4. Phase plot of system (1). (a) for $\beta = 0.002$ **. (b) for** $\beta = 0.005$**. (c) for** $\beta = 0.02$

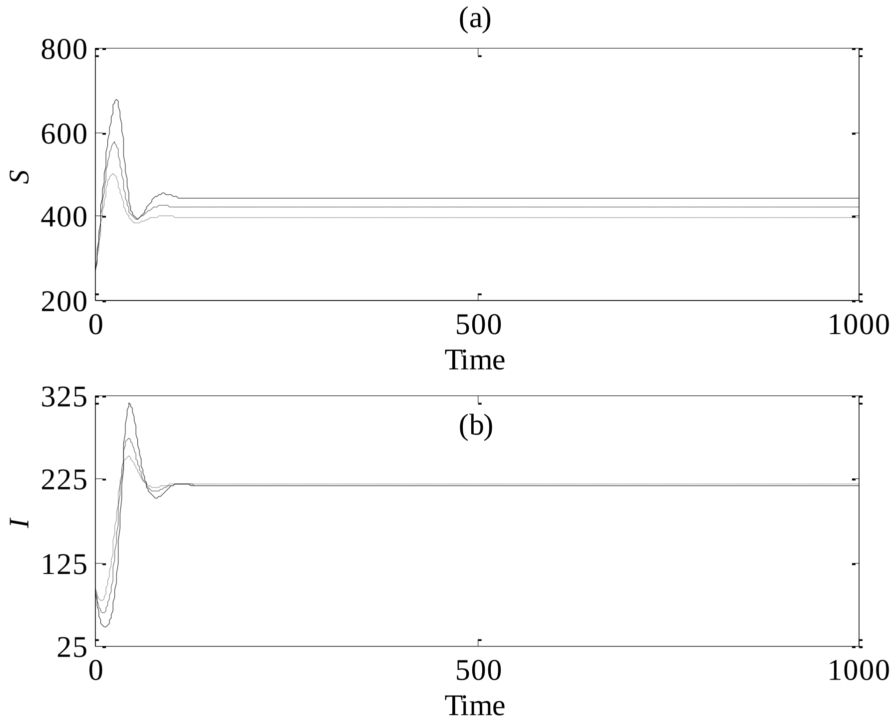

Fig. 5. Time series of the solutions of system (1); solid line for $p = 0.001$; dash line for $p = 0.5$; dot line for $p = 1$. (a) Trajectory of S as a function of time. (b) Trajectory of I as a function of time

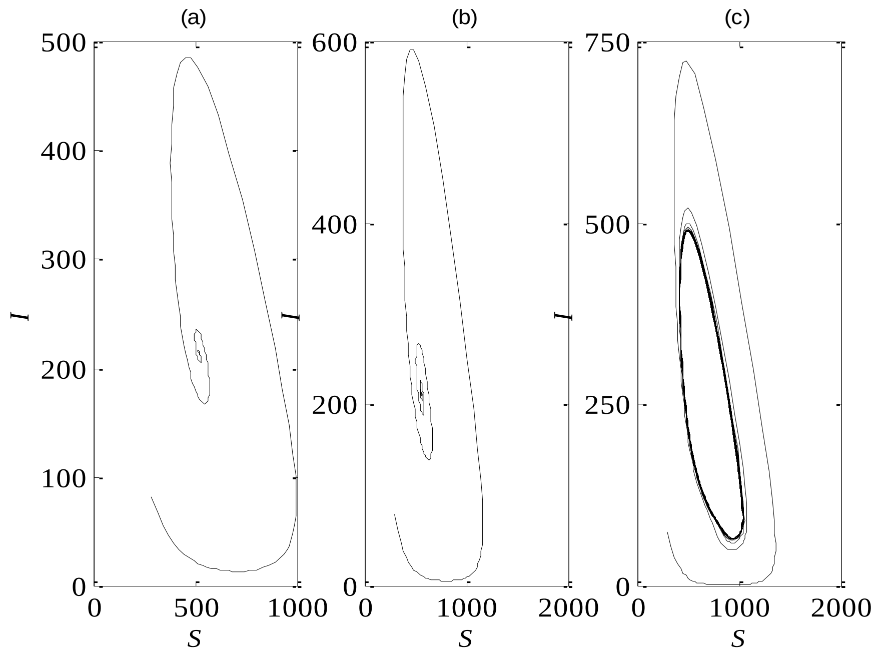

Fig. 6. Phase plot of system (1). (a) for $r = 3$. (b) for $r = 4$. (c) for $r = 5.24$

REFERENCES

Hale J.H. 1969. Ordinary differential equation, New York, Wiley-Interscience.

Hyman J.M. and Li J. 1998. Modeling the effectiveness of isolation strategies in preventing STD epidemics, *SIAM J. Appl. Math*., 58, 36-912.

Feng Z. and Thieme H.R. 1995. Recurrent outbreaks of childhood diseases revisited: the impact of isolation, *Math. Biosci*. 128, 93-115.

Gao L.Q., and Hethcote H.W. 1995. Disease models of with density-dependent demographics, *J. Math. Biol*. 50, 17-31.

Kermack W.O., Mckendrick A.G. 1927. Contributions to mathematical theory of epidemics, *Proc. R. Soc. Lond. A*., 115, 700-721.

Li J., and Ma Z., 2002. Qualitative analysis of *SIS*-epidemic model with vaccination and varying total population size, *Math. Comput. Model*, 20, 43-1235.

Li. X.Z., Li W.S. and Ghosh M. 2009. Stability and bifurcation of an *SIS*-epidemic model with treatment, *J. Chaos, Solution and Fractals*, 42, 2822-2832.

Wang W. 2006. Backward bifurcation of an epidemic model with treatment, *Math. Biosci*. 201(1): 58-71.

Zhou Y., Liu H. 2003. Stability of periodic solutions for an SIS model with pulse vaccination. *Math Comput Model* 38:299-308.

تأثير العلاج والمهجرين على ديناميكية النموذج الوبائي من النوع– SIS

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ملخــص

تم في هذا البحث، اقتراح وتحليل نموذج رياضي يمثل النموذج الوبائي من النوع – SIS والذي يتضمن مصدر علاج محدد والمهاجرين. افترضنا في النموذج اعلاه وجود جزء من المهاجرين العائدين مصابين. ناقشنا حدود النظام المقترح. بحثنا الاستقرارية المحلية لجميع نقاط التوازن الممكنة. كذلك درسنا الاستقرارية الشاملة للنظام تحليليا وعدديا. وأخيراً لاحظنا ان وجود العلاج والمهاجرين له تأثير ايجابي في استقرارية السلوك الديناميكي على النظام المقترح.

الكلمـــات الدالـــة: النموذج الوبائي من النوع – SIS، العلاج، المهاجرين، الاستقرارية المحلية، الاستقرارية الشاملة.

* قسم الرياضيات، كلية العلوم، جامعة بغداد، العراق. تاريخ استلام البحث 2011/11/2 وتاريخ قبوله 2012/4/10.