

Generalized Closed Fuzzy Sets: A Unified Approach

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ABSTRACT

Our system in this research depends on using the idea of the generalized closed sets in ordinary topological spaces for identification of the generalized closed fuzzy sets with its different types. Also trying for elucidation of the equivalent topological characters to transfer it to the fuzzy setting and elucidation of new results and theories, which we did not get before. We studied these types of the generalized closed fuzzy sets in one united study by entering, what we call, the $\varphi\psi$ - g -closed fuzzy sets where φ and ψ belong to the family $P = \{ \overset{\circ}{\cup}, \overset{\circ}{\cap} \}$ and represent the type of fuzzy closure operator and the fuzzy open set respectively. In similar way, we united the study of different notions of fuzzy mappings.

This study is characterized by the flexibility where it is possible to enter new types or characters for all those types in the future by using the five different types of fuzzy closure operators and open fuzzy sets.

Keywords: $\varphi\psi$ - g -closed, FR_1 -space, FR_2 -space, fuzzy φ - g -continuous, fuzzy φ - g -irresolute.

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1. INTRODUCTION

In 1970, Levine introduced the notion of generalized closed sets in topological spaces as a generalization of closed sets. Since then, many concepts related to generalized closed sets were defined and investigated. In 1997, Balasubramanian and Sundaram introduced the concepts of generalized closed sets in fuzzy setting. Also, they studied various generalizations fuzzy continuous mappings.

Recently, El-Shafei and Zakari (2005, 2007a&b) and Zakari (2005) introduced new types of generalized closed fuzzy sets in fuzzy topological spaces and studied many of their properties. Also, they studied various generalizations of fuzzy continuous mappings.

In the present paper, we introduce the concept of $\varphi\psi$ - g -closed, where φ represents a closure operator, and ψ represents a notion of generalized openness. By this way, we study each possible pairing of generalized closed

fuzzy sets in a unified way. Also, we introduce many of fuzzy generalized continuous mappings in a unified way.

2. PRELIMINARIES

Let X be a set and I the unit interval. A fuzzy set in X is an element of the set of all functions from X into I . The family of all fuzzy sets in X is denoted by I^X . A fuzzy singleton x_t is a fuzzy set in X defined by $x_t(x) = t, x_t(y) = 0$ for all $y \neq x, t \in (0, 1]$. The set of all fuzzy singletons in X is denoted by $S(X)$. For every $x_t \in S(X)$ and $\mu \in I^X$, we define $x_t \in \mu$ iff $t \leq \mu(x)$. A fuzzy set μ is called quasi-coincident with a fuzzy set ρ , denoted by $\mu q \rho$, iff there exists $x \in X$ such that $\mu(x) + \rho(x) > 1$. If μ is not quasi-coincident with ρ , then we write $\mu \bar{q} \rho$. By $cl(\mu), int(\mu), \mu^c, N(x_t, \tau)$ and $N_Q(x_t, \tau)$, we mean the fuzzy closure of μ , the fuzzy interior of μ the complement of μ , the class of all open neighborhoods of x_t and the class of all open Q -neighborhoods of x_t .

Definition 2.1 (Ganguly and Saha, 1986 and Mashhhour et al., 1982). A fuzzy subset μ of a fuzzy topological space (X, τ) is called:

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- (i) semi-open iff $\mu \leq \text{cl}(\text{int}(\mu))$.
- (ii) preopen iff $\mu \leq \text{int}(\text{cl}(\mu))$.
- (iii) α -open iff $\mu \leq \text{int}(\text{cl}(\text{int}(\mu)))$.

The complement of a semi-open (resp. preopen, α -open) fuzzy set is called a semi-closed (resp. preclosed, α -closed).

The α -closure of $\mu \in I^X$, denoted by $\text{cl}_\alpha(\mu)$, is the smallest α -closed fuzzy set containing μ .

Definition 2.2 (Ganguly and Saha, 1986&1988, Kandil, 1990). Let (X, τ) be a fuzzy topological space, $x_t \in S(X)$ and $\mu \in I^X$. Then:

- (i) The θ -closure of μ , denoted by $\text{cl}_\theta(\mu)$, is defined by:
 $x_t \in \text{cl}_\theta(\mu)$ iff $\text{cl}(\eta)q\mu$ for each $\eta \in N_Q(x_t, \tau)$.
- (ii) The δ -closure of μ , denoted by $\text{cl}_\delta(\mu)$, is defined by:
 $x_t \in \text{cl}_\delta(\mu)$ iff $\text{int}(\text{cl}(\eta))q\mu$ for each $\eta \in N_Q(x_t, \tau)$.
- (iii) The semi-closure of μ , denoted by $\text{cl}_s(\mu)$ is defined by:
 $x_t \in \text{cl}_s(\mu)$ iff $\eta q\mu$ for each $\eta \in N_Q(x_t, \text{SOF}(X))$.
 Where $\text{SOF}(X)$ denotes the class of all fuzzy semi-open sets in X .
- (iv) μ is called θ -closed (resp. δ -closed, semi-closed) iff $\mu = \text{cl}_\theta(\mu)$ (resp. $\mu = \text{cl}_\delta(\mu)$, $\mu = \text{cl}_s(\mu)$).

Definition 2.3 (kandil and El-Shafei, 1988). A fuzzy topological space (X, τ) is called:

- (i) FR_1 iff $x_t \bar{q} \text{cl}(y_r)$ implies that there exist $\eta \in N(x_t, \tau)$ and $\nu \in N(y_r, \tau)$ such that $\eta \bar{q} \nu$.
- (ii) FR_2 or F -regular iff $x_t \bar{q} \lambda$, λ is closed fuzzy set implies that there exist $\eta \in N(x_t, \tau)$ and $\nu \in \tau$, $\lambda \leq \nu$ such that $\eta \bar{q} \nu$.

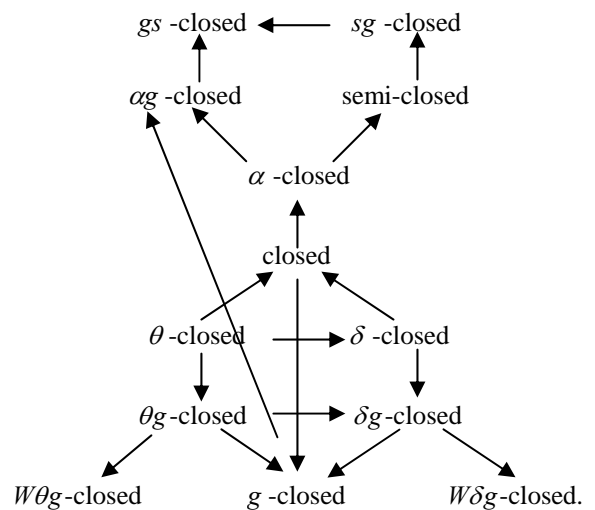
Definition 2.4 (Kandil et al., 1990). Let (X, τ) be a fuzzy topological space and $\mu \in I^X$. Then:

- (i) The family $\gamma = \{\eta_j : j \in J\} \subseteq \tau$ is called an open P -cover of μ iff for every $x_t \in \mu$ there exists $j_0 \in J$ such that $x_t \in \eta_{j_0}$.
- (ii) μ is called a C -set iff every open P -cover of μ has a finite subcover.

Definition 2.5 (Balasubramanian and Sundaram, 1997, El-Shafei and Zakari, 2005, 2007a&b, Zakari, 2005). Let (X, τ) be a fuzzy topological space. A fuzzy set $\mu \in I^X$ is called:

- (i) a generalized closed (g -closed, for short) iff $\text{cl}(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is open fuzzy set.
- (ii) a semi-generalized closed (sg -closed, for short) iff $\text{cl}_s(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is semi-open fuzzy set.
- (iii) a generalized semi-closed (gs -closed, for short) iff $\text{cl}_s(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is open fuzzy set.
- (iv) a α -generalized closed (αg -closed, for short) iff $\text{cl}_\alpha(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is open fuzzy set.
- (v) a θ -generalized closed (θg -closed, for short) iff $\text{cl}_\theta(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is open fuzzy set.
- (vi) a δ -generalized closed (δg -closed, for short) iff $\text{cl}_\delta(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is open fuzzy set.
- (vii) a weakly θ -generalized closed ($W\theta g$ -closed, for short) iff $\text{cl}_\theta(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is θ -open fuzzy set.
- (viii) a weakly δ -generalized closed ($W\delta g$ -closed, for short) iff $\text{cl}_\delta(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is δ -open fuzzy set.

The known relationships between the types of generalized closed fuzzy sets listed in the above definition can be summarized in the following diagram.



We address two general questions. Each generalization in the above definition involves a fuzzy closure operation

and a notion of "openness". Specifically, each definition involves either cl , cl_α , cl_s , cl_θ , cl_δ of $\mu \in I^X$ together with η being either open, α -open, semi-open, θ -open or δ -open. The first question, which arises from these definitions, is: do any new classes of generalized closed fuzzy sets exist if we consider every possible pairing of the five fuzzy closure operations mentioned above with the five notions of openness? In order to study each possible pairing in a unified way, we introduce the term $\varphi\psi$ - g -closed, where φ represents a fuzzy closure operation and ψ represents a notion of generalized openness. Surprisingly, in most cases, we obtain new characterizations of existing classes. These cases provide new insights into the nature of generalized closed fuzzy sets. The second question we consider is: are the implications represented in the diagram the only implications which apply in general? As a consequence of answering these two questions, we derive new relationships between different types of $\varphi\psi$ - g -closed fuzzy sets which characterize certain fuzzy topological spaces.

3. A unified approach: $\varphi\psi$ - g -closed fuzzy sets

In the following we denote closed (resp. semi-closed) by τ -closed (resp. s -closed), and $cl(\mu)$ by $cl_\tau(\mu)$ for $\mu \in I^X$, whenever it is convenient to do so. Similarly, we denote open (resp. semi-open) by τ -open (resp. s -open). Let $P = \{\tau, \alpha, s, \theta, \delta\}$.

Definition 3.1. Let (X, τ) be a fuzzy topological space and $\varphi, \psi \in P$. A fuzzy subset $\mu \in I^X$ is called $\varphi\psi$ - g -closed if $cl_\varphi(\mu) \leq \eta$ whenever $\mu \leq \eta$ and η is ψ -open.

Remark 3.2. Note that each type of generalized closed fuzzy sets in the Definition 2.5 is defined to be $\varphi\psi$ - g -closed for some $\varphi, \psi \in P$. Namely; a fuzzy set μ is:

- (i) g -closed iff it is $\tau\tau$ - g -closed.
- (ii) αg -closed iff it is $\alpha\tau$ - g -closed.
- (iii) sg -closed iff it is ss - g -closed.
- (iv) gs -closed iff it is $S\tau$ - g -closed.
- (v) θg -closed iff it is $\theta\tau$ - g -closed.
- (vi) $W\theta g$ -closed iff it is $\theta\theta$ - g -closed.
- (vii) δg -closed iff it is $\delta\tau$ - g -closed.

(viii) $W\delta g$ -closed iff it is $\delta\delta$ - g -closed.

Remark 3.3. We denote the class of all φ -open fuzzy sets on X by τ_φ . Also, we denote the class of all φ -open neighborhoods (resp. Q -neighborhoods) of x_i by $N(x_i, \tau_\varphi)$ (resp. $N_Q(x_i, \tau_\varphi)$), where $x_i \in S(X)$ and $\varphi \in P$.

Lemma 3.4. If (X, τ) is a fuzzy topological space, $\mu \in I^X$ and $\varphi \in \{\tau, \alpha, s\}$, then $x_i \in cl_\varphi(\mu)$ iff $\eta q \mu$ for each $\eta \in N_Q(x_i, \tau_\varphi)$.

Proof. Straightforward.

Lemma 3.5. If (X, τ) is a fuzzy topological space, $\mu \in I^X$, then:

- (i) $cl_\varphi(\mu) \leq cl(\mu)$, $\varphi \in \{\alpha, s\}$
- (ii) $cl(\mu) \leq cl_\varphi(\mu)$, $\varphi \in \{\theta, \delta\}$.

Proof. Straightforward.

Theorem 3.6. Let (X, τ) be a fuzzy topological space and $\varphi, \psi \in P$. Then every φs - g -closed fuzzy set is $\varphi\psi$ - g -closed.

Proof. Obvious.

The converse of Theorem 3.6 is not true, in general, by the following necessary counter example.

Example 3.7. Let $X = \{x, y\}$, $\tau = \{0_X, y_{0.8}, I_X\}$. If $\mu = 0.5$ and $\varphi = s$, $\psi = \theta$. Then μ is $\varphi\psi$ - g -closed, since the only ψ -open superset of μ is I_X . But μ is not φs - g -closed.

Theorem 3.8. Let (X, τ) be a fuzzy topological space such that for every $x_i \in S(X)$ either x_i is φ -open or x_i is ψ -closed. Then every $\varphi\psi$ - g -closed fuzzy subset of X is φ -closed for any $\varphi, \psi \in P$.

Proof. Suppose that for every $x_i \in S(X)$ either x_i is φ -open or x_i is ψ -closed and let $\mu \in I^X$ be $\varphi\psi$ - g -closed. If $x_i \bar{q} \mu$, we consider the following two cases:

Case 1. x_i is φ -open. Then x_i^c is φ -closed. Since $x_i \bar{q} \mu$, then $x_i \in \mu^c$. Since x_i^c is φ -closed, then $cl_\varphi(\mu) \leq cl_\varphi(x_i^c) = x_i^c$. This shows that $x_i \bar{q} cl_\varphi(\mu)$. Therefore μ is φ -closed.

Case 2. x_i is ψ -closed. Then x_i^c is ψ -open. Since $x_i \bar{q} \mu$, then $x_i \in \mu^c$. Since μ is $\varphi\psi$ - g -closed, then $cl_\varphi(\mu) \leq x_i^c$ and hence $x_i \bar{q} cl_\varphi(\mu)$. Therefore μ is φ -closed.

The converse of the above theorem is not true as we can see from the following example.

Example 3.9. Let $X = \{x\}$, $\tau = \{0_X, \{x_\lambda : \lambda \in (0, 0.2] \cup [0.7, 1]\}, I_X\}$. If $\varphi = \theta$, $\psi = \tau$, then every $\varphi\psi$ - g -closed fuzzy subset of X is φ -closed. But $x_{0.4}$ neither φ -open nor ψ -closed.

Theorem 3.10. Let (X, τ) be a fuzzy topological space such that for every $x_i \in S(X)$ either x_i is nowhere dense or preopen. Then a fuzzy subset μ of X is $\varphi\alpha$ - g -closed iff it is φs - g -closed, for any $\varphi \in \{\tau, \alpha, s\}$.

Proof. By Theorem 3.6, every φs - g -closed fuzzy set is $\varphi\alpha$ - g -closed for any $\varphi \in P$. To show the converse, let $\mu \in I^X$ be $\varphi\alpha$ - g -closed and $\mu \leq \eta$ where η is semi-open. If $x_i \bar{q} \eta$, then $\mu \leq \eta \leq x_i^c$. We consider the two cases:

Case 1. x_i is nowhere dense, then $cl(int(cl(x_i))) \leq x_i$ and hence x_i is α -closed and therefore x_i^c is α -open. Since μ is $\varphi\alpha$ - g -closed, we have $cl_\varphi(\mu) \leq x_i^c$. Thus $x_i \bar{q} cl_\varphi(\mu)$.

Case 2. x_i is preopen, then $x_i \leq int(cl(x_i))$. Since η is semi-open, we have $cl(\eta) = cl(int(\eta))$. Therefore we obtain the following:

$$cl_\varphi(\mu) \leq cl(\mu) \leq cl(\eta) = cl(int(\eta)) \leq cl(int(x_i^c)) = (int(cl(x_i)))^c, \text{ for any } \varphi \in \{\tau, \alpha, s\}.$$

So $cl_\varphi(\mu) \bar{q} int(cl(x_i))$ and hence $x_i \bar{q} cl_\varphi(\mu)$.

In either case, $cl_\varphi(\mu) \leq \eta$ and hence μ is φs - g -closed.

Theorem 3.11. Let (X, τ) be a fuzzy topological space, $\mu \in I^X$ and $\varphi \in P$, $\psi \in \{\tau, \theta, \delta\}$. Then μ is $\varphi\psi$ - g -closed iff for every $x_i \in S(X)$ such that $x_i q cl_\varphi(\mu)$ we have $cl_\psi(x_i) q \mu$.

Proof. Let $x_i q cl_\varphi(\mu)$ and suppose that $cl_\psi(x_i) \bar{q} \mu$. Since $cl_\psi(x_i)$ is ψ -closed fuzzy set, $\psi \in \{\tau, \theta, \delta\}$, then $(cl_\psi(x_i))^c \in \tau_\psi$. Since $\mu \leq (cl_\psi(x_i))^c$ and μ is $\varphi\psi$ - g -closed, then $cl_\varphi(\mu) \leq (cl_\psi(x_i))^c$ and so $cl_\psi(x_i) \bar{q} cl_\varphi(\mu)$ which implies that $x_i \bar{q} cl_\varphi(\mu)$. This is a contradiction.

Conversely, let η be ψ -open and $\mu \leq \eta$. If $x_i q cl_\varphi(\mu)$, then by assumption $cl_\psi(x_i) q \mu$. Hence there exists $y \in X$ such that $cl_\psi(x_i)(y) + \mu(y) > 1$. Put $cl_\psi(x_i)(y) = \varepsilon$. Then $y_\varepsilon \in cl_\psi(x_i)$ and $y_\varepsilon q \mu$. Thus $\rho q x_i$ for each $\rho \in N_Q(y_\varepsilon, \tau_\psi)$. Since $y_\varepsilon q \eta$, then $\eta q x_i$ and so $cl_\varphi(\mu) \leq \eta$. Thus μ is $\varphi\psi$ - g -closed.

Theorem 3.12. Let (X, τ) be a fuzzy topological space and let $\mu \in I^X$. Then μ is $\varphi\psi$ - g -closed if there is no

any non-empty ψ -closed fuzzy set λ such that $\lambda \bar{q} \mu$ and $\lambda q cl_\varphi(\mu)$, for any $\varphi \in P$, $\psi \in \{\tau, \theta, \delta\}$.

Proof. If μ is not $\varphi\psi$ - g -closed, then there exists ψ -open fuzzy set η such that $\mu \leq \eta$ and $cl_\varphi(\mu) \not\leq \eta$. Put $\lambda = \eta^c$, then λ is ψ -closed fuzzy set such that $\lambda \bar{q} \mu$ and $\lambda q cl_\varphi(\mu)$. This is a contradiction.

Conversely, suppose that there is a non-empty ψ -closed fuzzy set λ such that $\lambda \bar{q} \mu$ and $\lambda q cl_\varphi(\mu)$, since $\varphi \in P$, $\psi \in \{\tau, \theta, \delta\}$. Then there exists some $x_i \in \lambda$ such that $x_i q cl_\varphi(\mu)$. Since μ is $\varphi\psi$ - g -closed, then by Theorem 3.11, $cl_\psi(x_i) q \mu$ and hence $cl_\psi(\lambda) q \mu$. Since λ is ψ -closed, then we have $\lambda q \mu$. This is a contradiction.

Theorem 3.13. Let (X, τ) be FR_1 -space and let $\mu \in I^X$ be C -set in X . Then μ is $\varphi\psi$ - g -closed fuzzy set iff it is $\tau\psi$ - g -closed for any $\varphi \in \{\theta, \delta\}$, $\psi \in P$.

Proof. Put $\varphi = \theta$, and let (X, τ) be an FR_1 -space and μ is a C -set in X . We have to prove that $cl(\mu) = cl_\theta(\mu)$. Let $x_i \bar{q} cl(\mu)$. Since $\mu = \bigvee_{y_r \in \mu} y_r$, then

$$cl(\mu) = cl\left(\bigvee_{y_r \in \mu} y_r\right) \geq \bigvee_{y_r \in \mu} cl(y_r) \text{ and hence } x_i \bar{q} cl(y_r)$$

for every $y_r \in \mu$. Since (X, τ) is FR_1 -space, then there exist $\eta \in N(x_i, \tau)$ and $v \in N(y_r, \tau)$ such that $\eta \bar{q} v$ for every $y_r \in \mu$. The family $\{v : v \in N(y_r, \tau)\}$ is an open P -cover of μ . Since μ is C -set, then there exists a finite subcover, say, $\{v_i : i=1, \dots, n\}$. Moreover if

$$\rho = \bigvee_{i=1}^n v_i \text{ and } \sigma = \bigwedge_{i=1}^n \eta_i, \text{ then } \sigma \in N(x_i, \tau), \sigma \bar{q} \rho$$

and so $cl(\sigma) \bar{q} \rho$. Since $\mu \leq \rho$, we have $cl(\sigma) \bar{q} \mu$, $\sigma \in N(x_i, \tau)$ and hence $x_i \bar{q} cl_\theta(\mu)$. Therefore $cl(\mu) \leq cl_\theta(\mu)$. From Lemma 3.5(ii), we have $cl(\mu) = cl_\theta(\mu)$.

When $\varphi = \delta$, then the proof is similar to the above case.

Theorem 3.14. Let (X, τ) be FR_2 -space. Then $\mu \in I^X$ is $\varphi\psi$ - g -closed fuzzy set iff it is $\tau\psi$ - g -closed for any $\varphi \in \{\theta, \delta\}$ and any $\psi \in P$.

Proof. Put $\varphi = \theta$, and let (X, τ) be an FR_2 -space and $\mu \in I^X$. We have to prove that $cl(\mu) = cl_\theta(\mu)$. Let $x_i \bar{q} cl(\mu)$. Since (X, τ) is FR_2 -space and $cl(\mu)$ is closed, then there exist two open fuzzy sets η, v such

that $\eta \in N(x_i, \tau)$, $cl(\mu) \leq \nu$ and $\eta \bar{q} \nu$. So $\eta \leq \nu^c$ and hence $cl(\eta) \leq \nu^c$. This means that, there exists $\eta \in N(x_i, \tau)$ such that $cl(\eta) \bar{q} \nu$, i.e., there exists $\eta \in N(x_i, \tau)$ such that $cl(\eta) \bar{q} \mu$ and so $x_i \bar{q} cl_\theta(\mu)$. Therefore $cl_\theta(\mu) \leq cl(\mu)$. From Lemma 3.5(ii), we have $cl(\mu) = cl_\theta(\mu)$.

When $\varphi = \delta$, then the proof is similar to the above case.

Lemma 3.15. Let (X, τ) be a fuzzy topological space and $\mu, \eta \in I^X$. Then $cl_\varphi(\mu \vee \eta) = cl_\varphi(\mu) \vee cl_\varphi(\eta)$, for any $\varphi \in \{\tau, \theta, \delta\}$.

Proof. Straightforward.

Theorem 3.16. The finite union of $\varphi\psi$ -g-closed fuzzy sets is $\varphi\psi$ -g-closed, for any $\varphi \in \{\tau, \theta, \delta\}$ and any $\psi \in P$.

Proof. Suppose that $\mu, \eta \in I^X$ are $\varphi\psi$ -g-closed fuzzy sets such that $\varphi \in \{\tau, \theta, \delta\}$, $\psi \in P$ and let ν be ψ -open fuzzy set such that $\mu \vee \eta \leq \nu$. Then $cl_\varphi(\mu) \vee cl_\varphi(\eta) \leq \nu$. From Lemma 3.15, we have $cl_\varphi(\mu \vee \eta) \leq \nu$. Hence $\mu \vee \eta$ is $\varphi\psi$ -g-closed.

Lemma 3.17. Let μ be a preopen fuzzy subset of a fuzzy topological space (X, τ) . Then $cl_\varphi(\mu) = cl(\mu)$, for any $\varphi \in \{\alpha, \theta, \delta\}$.

Proof. Let μ be a preopen fuzzy set. We consider the following two cases:

Case 1. $\varphi = \alpha$. Then $cl_\alpha(\mu) = cl(\mu)$, since μ is closed iff μ is α -closed.

Case 2. $\varphi = \delta$. Let $x_i \bar{q} cl(\mu)$, then there exists $\nu \in N(x_i, \tau)$ such that $\nu \bar{q} \mu$, i.e. there exists $\nu \in N(x_i, \tau)$ such that $\nu \leq \mu^c$. But μ is preopen, so $cl(\nu) = cl(int(\nu)) \leq cl(int(\mu^c)) \leq \mu^c$. This implies that $cl(\nu) \bar{q} \mu$. Then $x_i \bar{q} cl_\theta(\mu)$. Therefore $cl_\theta(\mu) = cl(\mu)$.

When $\varphi = \delta$, then the proof is similar to the above case.

Theorem 3.18. Let μ be a preopen fuzzy subset of a fuzzy topological space (X, τ) . Then μ is $\tau\psi$ -g-closed fuzzy set iff it is $\varphi\psi$ -g-closed, for any $\varphi \in \{\alpha, \theta, \delta\}$ and any $\psi \in P$.

Proof. It is an immediate consequence of Lemma 3.17.

Lemma 3.19. Let (X, τ) be a fuzzy topological space. Then $cl_\varphi(\mu) = cl_\tau(\mu)$, for any $\mu \in \tau$ and any $\varphi \in \{\tau, \theta, \delta\}$.

Proof. Straightforward.

Theorem 3.20. Let (X, τ) be a fuzzy topological space and $\mu \in I^X$, $\varphi, \psi \in \{\tau, \theta, \delta\}$. Then the following conditions are equivalent:

- (i) μ is $\varphi\psi$ -g-closed.
- (ii) μ is ψ -open $\Rightarrow \mu$ is φ -closed.

Proof. (i) \Rightarrow (ii) Suppose that $\mu \in I^X$ is ψ -open fuzzy set. By (i), we have $cl_\varphi(\mu) \leq \mu$ and hence μ is φ -closed.

(ii) \Rightarrow (i) Suppose that $\mu \in I^X$. We have to prove that μ is $\varphi\psi$ -g-closed. Let $\mu \leq \eta$, since η ψ -open fuzzy subset of X . From (ii), η is φ -closed and hence τ -closed and τ -open. By using Lemma 3.19, we have $cl_\varphi(\mu) \leq cl_\varphi(\eta) = cl(\eta) = \eta$. Therefore μ is $\varphi\psi$ -g-closed.

4. A unified approach: $\varphi\psi$ -g-continuous mappings

Definition 4.1 (Balasubramanian and Sundaram, 1997). A fuzzy mapping $f : (X, \tau) \rightarrow (Y, \Delta)$ is called fuzzy generalized continuous (fuzzy g-continuous, for short) iff $f^{-1}(\eta)$ is g-closed in X for any closed fuzzy set η in Y .

Definition 4.2. A mapping $f : (X, \tau) \rightarrow (Y, \Delta)$ is called:

- (i) fuzzy $\varphi\psi$ -g-continuous iff the inverse image of every closed fuzzy set in Y is $\varphi\psi$ -g-closed in X .
- (ii) fuzzy $\varphi\psi$ -continuous iff the inverse image of every ψ -closed fuzzy set in Y is φ -closed in X .
- (iii) fuzzy $\varphi\psi$ -g-irresolute iff the inverse image of every $\varphi\psi$ -g-closed fuzzy set in Y is $\varphi\psi$ -g-closed in X .
- (iv) fuzzy φ -closed iff the image of every φ -closed fuzzy set in X is φ -closed in Y .
- (v) fuzzy φ -open iff the image of every φ -open fuzzy set in X is φ -open in Y .

Theorem 4.3. Every fuzzy $\varphi\psi$ -continuous mapping is fuzzy $\varphi\psi$ -g-continuous, $\varphi \in P$ and $\psi \in \{\tau, \alpha, s\}$.

Proof. Suppose that $\mu \in I^Y$ is closed fuzzy set. Then μ is ψ -closed for every $\psi \in \{\tau, \alpha, s\}$. Since f is fuzzy $\varphi\psi$ -continuous, then $f^{-1}(\mu)$ is φ -closed in X , $\varphi \in P$ and hence $f^{-1}(\mu)$ is $\varphi\psi$ -g-closed in X .

The converse of Theorem 4.3 is not true, in general, by the following necessary counterexample.

Example 4.4. Let $X = Y = \{x\}$ and let τ be a fuzzy topology in Example 3.11, $\Delta = \{0_Y, x_{0.8}, I_Y\}$. If $\varphi = s$, $\psi = \tau$, then the identity fuzzy mapping $f : (X, \tau) \rightarrow (Y, \Delta)$ is fuzzy $\varphi\psi$ - g -continuous but not fuzzy $\varphi\psi$ -continuous. Note that $\lambda = x_{0.2}$ is ψ -closed fuzzy set in Y but not φ -closed in X .

Theorem 4.5. Every fuzzy $\varphi\psi$ - g -irresolute mapping is fuzzy $\varphi\psi$ - g -continuous, for any $\varphi \in \{\tau, \alpha, s\}$ and any $\psi \in P$.

Proof. Suppose that $\mu \in I^Y$ be closed fuzzy set in Y . Then μ is $\varphi\psi$ - g -closed for any $\varphi \in \{\tau, \alpha, s\}$ and any $\psi \in P$. Since f is fuzzy $\varphi\psi$ - g -irresolute, then $f^{-1}(\mu)$ is $\varphi\psi$ - g -closed in X and hence f is fuzzy $\varphi\psi$ - g -continuous.

The converse of Theorem 4.5 is not true as we can see from the following example.

Example 4.6. If we consider the identity fuzzy mapping $f : (X, \tau) \rightarrow (Y, \Delta)$ of Example 4.4, then f is fuzzy $\varphi\psi$ - g -continuous but not fuzzy $\varphi\psi$ - g -irresolute. Note that $\lambda = x_{0.1}$ is $\varphi\psi$ - g -closed fuzzy set in Y but not $\varphi\psi$ - g -closed in X .

Theorem 4.7. If a fuzzy mapping $f : (X, \tau) \rightarrow (Y, \Delta)$ is bijective, fuzzy ψ -open and fuzzy $\varphi\psi$ - g -continuous, then f is fuzzy $\varphi\psi$ - g -irresolute for any $\varphi \in \{\tau, \theta, \delta\}$ and any $\psi \in P$.

Proof. Let λ be a $\varphi\psi$ - g -closed fuzzy set in Y and let $f^{-1}(\lambda) \leq \eta$, where η is ψ -open fuzzy set in X . Clearly, $\lambda \leq f(\eta)$. Since $f(\eta)$ is ψ -open fuzzy set in Y and λ is $\varphi\psi$ - g -closed, then $cl_\varphi(\lambda) \leq f(\eta)$ and thus $f^{-1}(cl_\varphi(\lambda)) \leq \eta$. Since f is fuzzy $\varphi\psi$ - g -continuous $cl_\varphi(\lambda)$ is φ -closed fuzzy set in Y , $\varphi \in \{\tau, \theta, \delta\}$, then $f^{-1}(cl_\varphi(\lambda))$ is $\varphi\psi$ - g -closed in X and hence

$cl_\varphi(f^{-1}(cl_\varphi(\lambda))) \leq \eta$. Thus $cl_\varphi(f^{-1}(\lambda)) \leq \eta$ and so $f^{-1}(\lambda)$ is a $\varphi\psi$ - g -closed in X . This shows that f is fuzzy $\varphi\psi$ - g -irresolute.

Theorem 4.8. If a fuzzy mapping $f : (X, \tau) \rightarrow (Y, \Delta)$ is fuzzy $\psi\psi$ -continuous and fuzzy φ -closed, $\mu \in I^X$ is $\varphi\psi$ - g -closed in X , then $f(\mu)$ is $\varphi\psi$ - g -closed in Y , for any $\varphi, \psi \in P$.

Proof. Let μ be $\varphi\psi$ - g -closed in X and $f(\mu) \leq \eta$, where η is ψ -open in Y . Since $\mu \leq f^{-1}(\eta)$, μ is $\varphi\psi$ - g -closed in X and since $f^{-1}(\eta)$ is ψ -open in X , then $cl_\varphi(\mu) \leq f^{-1}(\eta)$. Thus $f(cl_\varphi(\mu)) \leq \eta$. Hence $cl_\varphi(f(\mu)) \leq cl_\varphi(f(cl_\varphi(\mu))) = f(cl_\varphi(\mu)) \leq \eta$, since f is fuzzy φ -closed. Hence $f(\mu)$ is $\varphi\psi$ - g -closed in Y .

Theorem 4.9. Let $f : (X, \tau) \rightarrow (Y, \Delta)$ and $g : (Y, \Delta) \rightarrow (Z, \Omega)$ be two fuzzy mappings. Then:

- (i) $g \circ f$ is fuzzy $\varphi\psi$ - g -continuous, if g is fuzzy $\varphi\psi$ -continuous and f is fuzzy $\varphi\psi$ - g -continuous, for any $\varphi \in \{\tau, \theta, \delta\}$ and any $\psi \in \{\tau, \alpha, s\}$.
- (ii) $g \circ f$ is fuzzy $\varphi\psi$ - g -irresolute, if g is fuzzy $\varphi\psi$ - g -irresolute and f is fuzzy $\varphi\psi$ - g -irresolute, for any $\varphi, \psi \in P$.
- (iii) $g \circ f$ is fuzzy $\varphi\psi$ - g -continuous, if g is fuzzy $\varphi\psi$ - g -continuous and f is fuzzy $\varphi\psi$ - g -irresolute, for any $\varphi, \psi \in P$.

Proof. (i) Let $\mu \in I^Z$ be closed fuzzy subset. Then μ is ψ -closed for any $\psi \in \{\tau, \alpha, s\}$. Since g is fuzzy $\varphi\psi$ -continuous, then $g^{-1}(\mu)$ is φ -closed fuzzy set in Y and hence closed, for every $\varphi \in \{\tau, \theta, \delta\}$. Since f is fuzzy $\varphi\psi$ - g -continuous, then $f^{-1}(g^{-1}(\mu)) = (g \circ f)^{-1}(\mu)$ is $\varphi\psi$ - g -closed in X . Thus $g \circ f$ is fuzzy $\varphi\psi$ - g -continuous.

The prove of (ii) and (iii) are straightforward.

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$$P = \{ \tau, \alpha, s, \theta, \delta \} \quad \psi, \varphi \quad (\varphi\psi - \text{generalized closed fuzzy sets}) \quad \varphi\psi$$

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