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Reflection Principle for ∑-Biharmonic Functions

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ABSTRACT

In this paper, reflection principle for the class of elliptic partial differential equations with variable coefficients in the case of ∑-biharmonic functions is presented. Relations on reflection and some properties of ∑-Polyharmonic functions are found too.

Keywords*:* Reflection Principle, Harmonic Function, Biharmonic Function, Polyharmonic Function, Laplace Equation, Tricomi Equation, Reflection Principle, Heat Equation.

1. INTRODUCTION

The second-order partial differential equation: $\eta\frac{\partial^2 U}{\partial\zeta^2}+\frac{\partial^2 U}{\partial\eta^2}=0$, which is known as Tricomi's equation Çelebi (1968), is elliptic for $\eta>0.$ Using the transformation $x_1=\frac{2}{3}\eta^{3/2}, x_2=\zeta$, the canonical form of this equation is given by: $\frac{\partial^2 U}{\partial x_1^2}+\frac{\partial^2 U}{\partial x_2^2}+\frac{1}{3x_1}\frac{\partial U}{\partial x_1}=0$. In this paper, we investigate a class of partial differential equations with variable coefficients $L_{c,b,k}[U]=\sum_{i=1}^{n}c_i x_i^{2b_i}\left(\frac{\partial^2 U}{\partial x_i^2}+\frac{k_i}{x_i}\frac{\partial U}{\partial x_i}\right)=0$, where b_i, c_i and k_i are real constants with $c_i>0$, for $i=1,2,...,n$.

Equation $L_{c,b,k}[U]=0$ is one of the important type of the heat equation in $n-$ dimensional space which has been treated first by Rabadi (1983), and some types of solutions are found. Also some decomposition formulas of solutions are established in (Shawagfeh and Rabadi, 1989). Lord Kelvin established his principle for harmonic functions first in (1847), Çelebi (1968) established Lord Kelvin's principle for ∑-harmonic and ∑-polyharmonic functions of the generalized Tricomi's equation. Rabadi (2000) established Lord Kelvin's principle for the Equation (1) below. Finally (Abuarqob and Rabadi, 2005) gave some properties on the Lord Kelvin's principle for ∑-Polyharmonic functions.

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2. PRELIMINARIES

The objective of this paper is to extend reflection principle for ∑-biharmonic function and to present some relations on reflection principle of the equation:

$$L_{c,b,k}[U]=0, \tag{1}$$

where the operator $L_{c,b,k}$ is given by $L_{c,b,k}:\equiv\sum_{i=1}^{n}c_i x_i^{2b_i}\left(\frac{\partial^2}{\partial x_i^2}+\frac{k_i}{x_i}\frac{\partial}{\partial x_i}\right)$. We denote to the operator $L^{c,b,k}$ by $L^{c,b,k}:\equiv c_i x_i^{2b_i}\left(\frac{\partial^2}{\partial x_i^2}+\frac{k_i}{x_i}\frac{\partial}{\partial x_i}\right)$.

The domain of the operators $L_{c,b,k}$ and $L^{c,b,k}$ is the set of all real-valued functions

$U = U(x_1, x_2, ..., x_n) \in C^2(E)$ and E is the regularity domain of U in **R**. Through this paper we denote **R**: Set of real numbers and **N**: Set of natural numbers.

Definition (1): (Rabadi and Çelebi, 1986) The function U is called Σ-polyharmonic in a region E of the $n-$dimensional space, if $U \in C^{2p}(E)$ and satisfies the equation $L^p_{c,b,k}[U] = 0$, Σ-biharmonic if $U \in C^4(E)$ and satisfies the equation $L^2_{c,b,k}[U] = 0$, Σ-harmonic if $U \in C^2(E)$ and satisfies the equation $L_{c,b,k}[U] = 0$.

Remark (1): Equation (1) includes some equations as special cases:

1) If $c_i = 1, b_i = k_i = 0, i = 1, ..., n$, then

$L_{1,0,0}[U] = \sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i^2} = 0$ is called the Generalized Laplace equation.

2) If $c_i = 1, b_i = 0, i = 1, ..., n$, then

$L_{1,0,k}[U] = \sum_{i=1}^{n} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) = 0$ is called the Generalized Tricomi equation.

3. SOME PROPERTIES OF Σ-POLYHARMONIC FUNCTIONS

In this section, we present some useful properties of Σ-polyharmonic functions.

(1) **Lemma (1):** For $p, q \in \mathbf{N}$, then

$$L^p_{c,b,k}\left[\left(L^{c,b,k}\right)^q [U]\right] = \left(L_{c,b,k}\right)^q \left[L^p_{c,b,k}[U]\right].$$

Proof: It is enough to show that $L_{c,b,k}\left[L^{c,b,k}[U]\right] = L^{c,b,k}\left[L_{c,b,k}[U]\right]$ and the induction completes the proof.\

$$L_{c,b,k}\left[L^{c,b,k}[U]\right] =$$

$$\sum_{j=1}^{n} x_j^{2b_j} \left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) \left[x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) \right]$$

$$= \left\{ \sum_{j=1, j\neq i}^{n} x_j^{2b_j} \left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) + x_i^{2b_i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \right\} \left[x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) \right]$$

$$= x_i^{2b_i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \sum_{j=1, j\neq i}^{n} x_j^{2b_j} \left(\frac{\partial^2 U}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial U}{\partial x_j} \right) + x_i^{2b_i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \left[x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) \right]$$

$$= x_i^{2b_i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \left\{ \begin{array}{l} \sum_{j=1, j\neq i}^{n} x_j^{2b_j} \left(\frac{\partial^2 U}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial U}{\partial x_j} \right) \\ + x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) \end{array} \right\}$$

$$= L^{c,b,k}\left[L_{c,b,k}[U]\right]$$

Theorem (1): Let $U \in C^{2p+1}(E)$. Then, for any natural number p

$$L^p_{c,b,k}\left[x_i \frac{\partial U}{\partial x_i} \right] = x_i \frac{\partial}{\partial x_i} L^p_{c,b,k}[U] + 2p(1 - b_i) L^{c,b,k}\left[L^{p-1}_{c,b,k}[U]\right]$$

Proof:

$$L_{c,b,k}\left[x_i \frac{\partial U}{\partial x_i} \right] = \left\{ \begin{array}{l} \sum_{j=1, j\neq i}^{n} x_j^{2b_j} \left(\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial}{\partial x_j} \right) \\ + x_i^{2b_i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \end{array} \right\} \left[x_i \frac{\partial U}{\partial x_i} \right]$$

$$= x_i \frac{\partial}{\partial x_i} \sum_{j=1, j\neq i}^{n} x_j^{2b_j} \left(\frac{\partial^2 U}{\partial x_j^2} + \frac{k_j}{x_j} \frac{\partial U}{\partial x_j} \right)$$

$$+ x_i^{2b_i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \left[x_i \frac{\partial U}{\partial x_i} \right]$$

But

$$\frac{\partial}{\partial x_i} \left[x_i \frac{\partial U}{\partial x_i} \right] = x_i \frac{\partial^2 U}{\partial x_i^2} + \frac{\partial U}{\partial x_i},$$

$$\frac{\partial^2}{\partial x_i^2} \left[x_i \frac{\partial U}{\partial x_i} \right] = x_i \frac{\partial^3 U}{\partial x_i^3} + 2 \frac{\partial^2 U}{\partial x_i^2}.$$

Now, we have

$$x_i^{2b_i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \left[x_i \frac{\partial U}{\partial x_i} \right]$$

$$= x_i^{2b_i} \left\{ x_i \frac{\partial}{\partial x_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) + 2 \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) \right\}$$

$$= x_i \frac{\partial}{\partial x_i} \left\{ x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) \right\} - 2b_i x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right)$$

$$+ 2x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right)$$

$$= x_i \frac{\partial}{\partial x_i} \left\{ x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) \right\}$$

$$- 2(1 - b_i) x_i^{2b_i} \left(\frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right).$$

Hence

$$L_{c,b,k} \left[x_i \frac{\partial U}{\partial x_i} \right] = x_i \frac{\partial}{\partial x_i} L_{c,b,k}[U] + 2(1 - b_i) L^{c,b,k}[U]$$

Also by the same arguments, we have

$$L_{c,b,k} \left[L_{c,b,k} \left[x_i \frac{\partial U}{\partial x_i} \right] \right]$$

$$= L_{c,b,k} \left[x_i \frac{\partial}{\partial x_i} L_{c,b,k}[U] \right] + 2(1 - b_i) L^{c,b,k} \left[L_{c,b,k}[U] \right]$$

Now,

$$L_{c,b,k} \left[x_i \frac{\partial}{\partial x_i} L_{c,b,k}[U] \right]$$

$$= x_i \frac{\partial}{\partial x_i} L^2_{c,b,k}[U] + 2(1 - b_i) L^{c,b,k} \left[L_{c,b,k}[U] \right]$$

and

$$L^2_{c,b,k} \left[x_i \frac{\partial U}{\partial x_i} \right]$$

$$= x_i \frac{\partial}{\partial x_i} L^2_{c,b,k}[U] + 4(1 - b_i) L^{c,b,k} \left[L_{c,b,k}[U] \right] \tag{2}$$

Replace U by $L^{\alpha}_{c,b,k}[U]$ in Equation (2) to get

$$L_{c,b,k} \left[x_i \frac{\partial}{\partial x_i} L^{\alpha}_{c,b,k}[U] \right].$$

$$= x_i \frac{\partial}{\partial x_i} L^{\alpha+1}_{c,b,k}[U] + 2(1 - b_i) L^{c,b,k} \left[L^{\alpha}_{c,b,k}[U] \right]$$

Assume that Theorem (1) holds for $p = \alpha$, then we have

$$L^{\alpha}_{c,b,k} \left[x_i \frac{\partial U}{\partial x_i} \right]$$

$$= x_i \frac{\partial}{\partial x_i} L^{\alpha}_{c,b,k}[U] + 2\alpha(1 - b_i) L^{c,b,k} \left[L^{\alpha-1}_{c,b,k}[U] \right] \tag{3}$$

Applying the operator $L_{c,b,k}$ to both sides of Equation (3), we get

$$L^{\alpha+1}_{c,b,k} \left[x_i \frac{\partial U}{\partial x_i} \right]$$

$$= L_{c,b,k} \left[x_i \frac{\partial}{\partial x_i} L^{\alpha}_{c,b,k}[U] \right] + 2\alpha(1 - b_i) L^{c,b,k} \left[L^{\alpha}_{c,b,k}[U] \right]$$

and then

$$L^{\alpha+1}_{c,b,k} \left[x_i \frac{\partial U}{\partial x_i} \right]$$

$$= x_i \frac{\partial}{\partial x_i} L^{\alpha+1}_{c,b,k}[U] + 2(\alpha + 1)(1 - b_i) L^{c,b,k} \left[L^{\alpha}_{c,b,k}[U] \right],$$

this completes the proof.

Remark (2): Some results related to the ∑-harmonic functions (Rabadi and Çelebi, 1986)

1) $$L^p_{c,b,k} \left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right)^q$$

$$= \left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right)^q L^p_{c,b,k}, \quad c_i = 2p(1 - b_i) + b_i, \; i = 1, 2, \ldots, n, \; p, q \in \mathbf{N}.$$

2) $L_{c,b,k}\left(\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right)$

$$= \left(2+\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right) L_{c,b,k} - 2\sum_{i=1}^{n} b_i x_i^{2b_i}\left(\frac{\partial^2}{\partial x_i^2}+\frac{k_i}{x_i}\frac{\partial}{\partial x_i}\right)$$

, $b_i \neq 1,0,\ i=1,2,...,n$.

Now, we want to reproduce Σ-polyharmonic functions of order p from a given Σ-polyharmonic function of the same order. We denote the solution of equation $L^{p}_{c,0,k}[U]=0$ by $u_p\{k_1,...,k_n\}$ and $U^{*}_{p}(E)$ be the set of all solutions of $L^{p}_{c,0,k}[U]=0$ in the domain E.

Lemma (2): Let $u_p\{k_1,...,k_n\}\in U^{*}_{p}(E)$. Then $x_i^{p+s} L^{s}_{c,0,k}\left[\frac{u_p}{x_i^{p-s}}\right]$ is Σ-polyharmonic function of order p, where $x_i \neq 0,\ p>s$ and $p,s\in \mathbf{N}$.

Proof: Let $W\in C^2(E)$, then

$$x_i^3 L_{c,0,k}\left[\frac{W}{x_i}\right] = (2-k_i)c_i W + x_i^2 L_{c,0,k}[W] - 2c_i x_i \frac{\partial W}{\partial x_i} \tag{4}$$

Applying $L^2_{c,0,k}$ to both sides of Equation (4), we get

$$L^2_{c,0,k}\left[x_i^3 L_{c,0,k}\left[\frac{W}{x_i}\right]\right] = (2-k_i)c_i L^2_{c,0,k}[W] + L^2_{c,0,k}\left[x_i^2 L_{c,0,k}[W]\right]. \tag{5}$$

$$-2c_i L^2_{c,0,k}\left[x_i \frac{\partial W}{\partial x_i}\right]$$

Now we can obtain the following results:

- $L^2_{c,0,k}\left[x_i^2 L_{c,0,k}[W]\right]$

$$= L_{c,0,k}\left[x_i^2 L^2_{c,0,k}[W]\right] + 4c_i L_{c,0,k}\left[x_i \frac{\partial}{\partial x_i} L_{c,0,k}[W]\right] + (2+2k_i)c_i L^2_{c,0,k}[W]$$

- $L_{c,0,k}\left[x_i^2 L^2_{c,0,k}[W]\right]$

$$= x_i^2 L^3_{c,0,k}[W] + 4c_i x_i \frac{\partial}{\partial x_i} L^2_{c,0,k}[W] + (2+2k_i)c_i L^2_{c,0,k}[W]$$

- $L_{c,0,k}\left[x_i \frac{\partial}{\partial x_i} L_{c,0,k}[W]\right]$

$$= x_i \frac{\partial}{\partial x_i} L^2_{c,0,k}[W] + 2L^{c,b,k}\left[L_{c,0,k}[W]\right]$$

- $L^2_{c,0,k}\left[x_i \frac{\partial W}{\partial x_i}\right]$

$$= L_{c,0,k}\left[x_i \frac{\partial}{\partial x_i} L_{c,0,k}[W]\right] + 2L^{c,b,k}\left[L_{c,0,k}[W]\right]$$

Substituting the above results in Equation (5), we get

$$L^2_{c,0,k}\left[x_i^3 L_{c,0,k}\left[\frac{W}{x_i}\right]\right] = \left\{\frac{3}{1}(2+k_i)c_i + x_i^2 L_{c,0,k} + 6c_i x_i \frac{\partial}{\partial x_i}\right\} L^2_{c,0,k}[W] \tag{6}$$

From Equation (6), if $W\in U^{*}_{2}(E)$, then $x_i^3 L_{c,0,k}\left[\frac{W}{x_i}\right]\in U^{*}_{2}(E)$. This proves the lemma for $s=1,\ p=1$.

Now, if

$$x_i^p L_{c,0,k}\left[\frac{W}{x_i^{p-2}}\right] = (p-2)(p-1-k_i)c_i W + x_i^2 L_{c,0,k}[W] - 2(p-2)c_i x_i \frac{\partial W}{\partial x_i}$$

, then

$$x_i^{p+1} L_{c,0,k}\left[\frac{W}{x_i^{p-1}}\right] = x_i^{p+1} L_{c,0,k}\left[\frac{1}{x_i^{p-2}}\left(\frac{W}{x_i}\right)\right]$$

$$= x_i^{p+1}\left\{\begin{array}{l}\frac{(p-2)(p-1-k_i)c_i}{x_i^{p+1}}W \quad -\frac{2(p-2)c_i x_i}{x_i^p}\left(\frac{1}{x_i}\frac{\partial W}{\partial x_i}-\frac{1}{x_i^2}W\right)\\ +\frac{x_i^2}{x_i^p}L_{c,0,k}\left[\frac{W}{x_i}\right]\end{array}\right\}$$

$$= (p-2)(p-1-k_i)c_i W + x_i^3 L_{c,0,k}\left[\frac{W}{x_i}\right] - 2(p-2)c_i x_i\frac{\partial W}{\partial x_i} + 2(p-2)c_i W.$$

Using Equation (4), we have

$$x_i^{p+1}L_{c,0,k}\left[\frac{W}{x_i^{p-1}}\right] = (p-1)(p-k_i)c_i W + x_i^2 L_{c,0,k}[W] - 2(p-1)c_i x_i\frac{\partial W}{\partial x_i} \quad (7)$$

So, we can show that $x_i^{p+1}L_{c,0,k}\left[\frac{W}{x_i^{p-1}}\right] \in U_p^*(E)$ by applying the operator $L^p_{c,0,k}$ to both sides of Equation (7), to obtain

$$L^p_{c,0,k}\left[x_i^{p+1}L_{c,0,k}\left[\frac{W}{x_i^{p-1}}\right]\right] = (p-1)(p-k_i)c_i L^p_{c,0,k}[W] - 2(p-1)c_i \left\{\frac{2}{1}pL^{c,b,k}\left[L^{p-1}_{c,0,k}[W]\right] + x_i\frac{\partial}{\partial x_i}L^p_{c,0,k}[W]\right\}$$
$$+ L^{p-1}_{c,0,k}\left[\begin{array}{l}2(1+k_i)c_i L_{c,0,k}[W]\\ +4c_i x_i\frac{\partial}{\partial x_i}L_{c,0,k}[W] + x_i^2 L^2_{c,0,k}[W]\end{array}\right].$$

Assume $W \in U_P^*(E)$, then

$$L^p_{c,0,k}\left[x_i^{p+1}L_{c,0,k}\left[\frac{W}{x_i^{p-1}}\right]\right] = -4p(p-1)c_i L^{c,b,k}\left[L^{p-1}_{c,0,k}[W]\right] \quad (8)$$
$$+4c_i L^{p-1}_{c,0,k}\left[x_i\frac{\partial}{\partial x_i}L_{c,0,k}[W]\right] + L^{p-1}_{c,0,k}\left[x_i^2 L^2_{c,0,k}[W]\right].$$

Now we obtain the following results:

- $L^{p-1}_{c,0,k}\left[x_i\frac{\partial}{\partial x_i}L_{c,0,k}[W]\right]$
- $= x_i\frac{\partial}{\partial x_i}L^p_{c,0,k}[W] + 2(p-1)L^{c,0,k}\left[L^{p-1}_{c,0,k}[W]\right]$.

$$L^{p-1}_{c,0,k}\left[x_i^2 L^2_{c,0,k}[W]\right]$$
$$= (2+2k_i)c_i L^p_{c,0,k}[W] + L^{p-2}_{c,0,k}\left[x_i^2 L^3_{c,0,k}[W]\right] + 4c_i L^{p-2}_{c,0,k}\left[x_i\frac{\partial}{\partial x_i}L^2_{c,0,k}[W]\right]$$

- $L^{p-2}_{c,0,k}\left[x_i\frac{\partial}{\partial x_i}L^2_{c,0,k}[W]\right]$

$$= x_i\frac{\partial}{\partial x_i}L^p_{c,0,k}[W] + 2(p-2)L^{c,0,k}\left[L^{p-1}_{c,0,k}[W]\right]$$

- $L^p_{c,0,k}[W] = 0$.

Substituting these results in Equation (8), we get

$$L^p_{c,0,k}\left[x_i^{p+1}L_{c,0,k}\left[\frac{W}{x_i^{p-1}}\right]\right] = \left(-4p(p-1) + 8\{(p-1)+(p-2)+\ldots+1\}\right). c_i L^{c,0,k}\left[L^{p-1}_{c,0,k}[W]\right] = 0$$

in the other words, if $W \in U_p^*(E)$, then there exists $u_p\{k_1,\ldots,k_n\} \in U_p^*(E)$, such that

$$x_i^{p+1}L_{c,0,k}\left[\frac{W}{x_i^{p-1}}\right] = u_p\{k_1,\ldots,k_n\}.$$

Now we want to show that $x_i^{p+s}L^s_{c,0,k}\left[\frac{W}{x_i^{p-s}}\right]$ is a Σ-polyharmonic function of order p. For $s = 2$; we have the following

$$x_i^{p+2}L^2_{c,0,k}\left[\frac{W}{x_i^{p-2}}\right] = x_i^{p+2}L_{c,0,k}\left[\frac{V}{x_i^p}\right], \quad \text{where}$$

$V = x_i^p L_{c,0,k}\left[\frac{W}{x_i^{p-2}}\right]$. Since V is Σ-polyharmonic function of order $p-1$, then $V \in U_p^*(E)$ and hence there exists $u_p\{k_1,\ldots,k_n\} \in U_p^*(E)$ such that

$$x_i^{p+2}L_{c,0,k}\left[\frac{V}{x_i^p}\right] = x_i^{p+2}L^2_{c,0,k}\left[\frac{W}{x_i^{p-2}}\right] = u_p\{k_1,\ldots,k_n\}$$

Assume that there exists $u_p\{k_1,\ldots,k_n\} \in \mathbf{U}_p^*(E)$, such that $x_i^{p+\gamma} L_{c,0,k}^{\gamma}\left[\frac{W}{x_i^{p-\gamma}}\right] = u_p\{k_1,...,k_n\}, \quad \gamma \in \mathbf{N}$. Then we get

$$x_i^{p+\gamma+1} L_{c,0,k}^{\gamma+1}\left[\frac{W}{x_i^{p-\gamma-1}}\right]$$

$$= x_i^{p+\gamma+1} L_{c,0,k}\left[\left(x_i^{p+\gamma-1} L_{c,0,k}^{\gamma}\left[\frac{W}{x_i^{p-\gamma-1}}\right]\right) \Big/ x_i^{p+\gamma-1}\right]$$

Let $F :\equiv x_i^{p+\gamma-1} L_{c,0,k}^{\gamma}\left[\frac{W}{x_i^{p-\gamma-1}}\right]$. Then, clearly $F \in \mathbf{U}_p^*(E)$. Hence there exists a function $u_p\{k_1,...,k_n\} \in \mathbf{U}_p^*(E)$ such that $F = u_p\{k_1,...,k_n\}$ for $p-\gamma-1>0$ and $W \in \mathbf{U}_p^*(E)$. This completes the proof.

Remark (3): Some historical notes:

1) Lemma (2) for $p \le s$, is established in the case $c_i = 1,\ b_i = 0,\ i = 1,2,...,n$, by (Çelebi, 1968).

2) Theorem (1) and Lemma (2) are established by (Rabadi and Çelebi, 1986), for the case $c_i = 1,\ b_i = 0,$ for $i = 1,2,...,n$.

4. REFLECTION PRINCIPLE FOR ∑-BIHARMONIC FUNCTIONS

In this section, we consider the following notations in $\mathbf{R}^n$: $\sim \overline{H}$ is the negation of the closure of the set H, $H_i = \{p : x_i > 0\}, D_i = \{p : x_i = 0\}$, $C_r = \{p : x_1^2 + x_2^2 + ... + x_n^2 < r^2\}$, where $p \in \mathbf{R}^n$ and $r \in \mathbf{R}$

Theorem (2) (Rabadi and Çelebi, 1986) Let $u_2\{k_1,k_2\} \in U_2^*(C_r \cap H)$. If $u_2\{k_1,k_2\}$ satisfies the conditions: **(a)** $\lim_{x_1\to 0} x_1^{k_1} u_2\{k_1,k_2\} = 0$, for $k_1 > 0$ or

(b) $\lim_{x_1\to 0} u_2\{k_1,k_2\} = 0$, for $k_1 < 0$, in the domain $S \subset D$, then $u_2\{k_1,k_2\}$ can be extended continuously to the domain $C_r \cap (\sim \overline{H})$ as a ∑-biharmonic function in the form: $u_2^*(-x_1,x_2) = -u_2(x_1,x_2) +$

$$4(6-k_1-k_2)x_1^{1-k_1}x_2^{1-k_2} + 4(5-k_1)x_1^{3-k_1}x_2^{1-k_2} + 4(5-k_2)x_1^{1-k_1}x_2^{3-k_2}$$

$$+ L_{c,0,k}\left[(-x_1)^{1-k_1}x_2^{1-k_2}\{x_1^4 + x_2^4 + x_1^2x_2^2 + x_1^2 + x_2^2\}\right]$$

Theorem (3): Let $u_2\{k_1,...,k_n\} \in U_2^*(C_r \cap H_i)$, and $L_{c,0,k-2}[u_2(p)] + 2\sum_{j=1, j\ne i}^{n} \frac{c_j}{x_j}\frac{\partial u_2(p)}{\partial x_j} = 0$. If the function $u_2\{k_1,...,k_n\}$ satisfies the conditions:

(a) $\lim_{p\to p_0} x_i^{1-k_i} u_2\{k_1,...,k_n\} = 0$, $k_i > 0$ or

(b) $\lim_{p\to p_0} x_i^{k_i-1} u_2\{k_1,...,k_n\} = 0$, $k_i < 0$ for $p_0 \in S \subset D_i$, then $u_2\{k_1,...,k_n\}$ can be extended continuously to the domain $C_r \cap (\sim \overline{H_i})$ as a ∑-biharmonic function in the form:

$$u_2^*(p^*) = \left\{u_2(p) - x_i^3 L_{c,0,k}\left[\frac{u_2(p)}{x_i}\right]\right\} \Big/ \{(2-k_i)c_i - 1\}$$

, $k_i \ne 2 - \frac{1}{c_i}, i = 1,...,n$, where p^* is the reflection of p with respect to D_i.

Proof: We should show the following:

(1) $u_2^*(p^*)$ is ∑-biharmonic function.

(2) $$u_2(p) = \left\{u_2^*(p^*) - x_i^3 L_{c,0,k}\left[\frac{u_2^*(p^*)}{x_i}\right]\right\} \Big/ \{(2-k_i)c_i - 1\} \tag{9}$$

(3) $$U(x_1,x_2,...,x_n) = \begin{cases} u_2(p), & p \in C_r \cap H_i, \\ u_2^*(p^*), & p^* \in C_r \cap (\sim \overline{H_i}) \end{cases}$$, is

continuous on D_i

Since

$$L^2_{c,0,k}\left[u_2^*\left(p^*\right)\right] = \left\{ L^2_{c,0,k}\left[u_2(p)\right] - L^2_{c,0,k}\left[x_i^3 L_{c,0,k}\left[\frac{u_2(p)}{x_i}\right]\right]\right\} \Big/ \left\{(2-k_i)c_i - 1\right\},$$

and by Lemma (2), we get $L^2_{c,0,k}\left[u_2^*\left(p^*\right)\right] = 0$. So, $u_2^*\left(p^*\right)$ is Σ-biharmonic function.

For the second statement we substitute Equation (8) in the right hand side of Equation (9), to get the following

$$\left\{u_2^*\left(p^*\right) - x_i^3 L_{c,0,k}\left[\frac{u_2^*\left(p^*\right)}{x_i}\right]\right\} \Big/ \left\{(2-k_i)c_i - 1\right\}$$

$$= \left\{u_2(p) - 2x_i^3 L_{c,0,k}\left[\frac{u_2(p)}{x_i}\right] + x_i^3 L_{c,0,k}\left[x_i^2 L_{c,0,k}\left[\frac{u_2(p)}{x_i}\right]\right]\right\} \Big/ \left\{(2-k_i)c_i - 1\right\}^2. \qquad (10)$$

Now we obtain the following results:

- $$x_i^3 L_{c,0,k}\left[x_i^2 L_{c,0,k}\left[\frac{u_2(p)}{x_i}\right]\right] = x_i^3 L_{c,0,k}\left[x_i L_{0,k}\left[u_2(p)\right]\right] - 2c_i x_i^3 L_{c,0,k}\left[\frac{\partial u_2(p)}{\partial x_i}\right] + (2-k_i)c_i x_i^3 L_{c,0,k}\left[\frac{u_2(p)}{x_i}\right].$$
- $$x_i^3 L_{c,0,k}\left[\frac{u_2(p)}{x_i}\right] = x_i^2 L_{c,0,k}\left[u_2(p)\right] - 2c_i x_i \frac{\partial u_2(p)}{\partial x_i} + (2-k_i)c_i u_2(p)$$
- $$x_i^3 L_{c,0,k}\left[x_i L_{c,0,k}\left[u_2(p)\right]\right] = x_i^4 L^2_{c,0,k}\left[u_2(p)\right] + 2c_i x_i^3 \frac{\partial}{\partial x_i} L_{c,0,k}\left[u_2(p)\right] + c_i k_i x_i^2 L_{c,0,k}\left[u_2(p)\right].$$
- $$L_{c,0,k}\left[\frac{\partial u_2(p)}{\partial x_i}\right] - \frac{\partial}{\partial x_i} L_{c,0,k}\left[u_2(p)\right] = \frac{c_i k_i}{x_i^2}\frac{\partial u_2(p)}{\partial x_i}.$$

Substituting these results in Equation (10), we obtain

$$\left\{u_2^*\left(p^*\right) - x_i^3 L_{c,0,k}\left[\frac{u_2^*\left(p^*\right)}{x_i}\right]\right\} \Big/ \left\{(2-k_i)c_i - 1\right\}$$

$$= \left\{\left\{(2-k_i)c_i - 1\right\}^2 \frac{u_2(p)}{1} - 2(1-c_i)x_i^2 L_{c,0,k}\left[u_2(p)\right] + 4c_i(1-c_i)x_i \frac{\partial u_2(p)}{\partial x_i}\right\} \Big/ \left\{(2-k_i)c_i - 1\right\}^2$$

$$= u_2(p) - 2(1-c_i)x_i^2\left(L_{c,0,k-2} + 2\sum_{j=1, j\neq i}^{n} \frac{c_j}{x_j}\frac{\partial}{\partial x_j}\right)\left[u_2(p)\right] = u_2(p).$$

For the third statement, assume that $p \in C_r \cap H_i$ and $p_0 \in S$. Then, it is clear that

(a) $\lim\limits_{p \to p_0} x_i^{1-k_i} u_2^*\left\{k_1, \ldots, k_n\right\} = 0$, for $k_i > 0$, or

(b) $\lim\limits_{p \to p_0} x_i^{k_i - 1} u_2^*\left\{k_1, \ldots, k_n\right\} = 0$, for $k_i < 0$. This completes the proof.

Remark (4): Some special case:

1) For $c_j = 1, k_i = 2, k_j = 0, j \neq i, j = 1, \ldots, n$, Theorem (3) is obtained by (Rabadi, Çelebi, 1986).

2) For $b_i = 0, c_i = 1, i = 1, \ldots, n$, Theorem (3) is established by (Rabadi, Çelebi, 1986).

5. APPLICATION ON THE HEAT EQUATION

In this section we utilize an application on equation $L_{c,b,k}[U] = 0$. To see this consider the solution $W\left(x_1, \ldots, x_5, 0, \ldots, 0\right) = e^{-s/4\beta} U\left(t, x_3, x_4, x_5, 0, \ldots, 0\right)$, t

$$= x_1 + ix_2, s = x_1 - ix_2, i^2 = -1, \beta \in \mathbf{R}^+ \qquad (11)$$

of equation

$$L_{c,b,k}[W] = 4\frac{\partial^2 W}{\partial s \partial t} + \sum_{j=3}^{n} x_i^{2b_i}\left(\frac{\partial^2 W}{\partial x_j^2} + \frac{k_j}{x_j}\frac{\partial W}{\partial x_j}\right) = 0 \qquad (12)$$

Computing the derivatives and substituting it in $L_{c,b,k}[W]=0$, we obtain the heat equation:

$$\sum_{j=3}^{n} x_i^{2b_i}\left(\frac{\partial^2 U}{\partial x_j^2}+\frac{k_j}{x_j}\frac{\partial U}{\partial x_j}\right)=\frac{1}{\beta}\frac{\partial U}{\partial t} \qquad (13)$$

That is, if $U=U(t,x_3,x_4,x_5,0,...,0)$ is a solution of Equation (13), then Equation (11) is a solution of Equation (12).

6. CONCLUSIONS

We present some properties of ∑-polyharmonic functions and extended reflection principle for ∑-biharmonic function for a class of fourth order elliptic partial differential equations with variable coefficients. In the future research we try to extend the results in section (5) of this paper under the condition: $b_i \in \mathbf{R}$, for $i=1,2,...,n$.

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مبدأ الانعكاس للاقترانات ∑ التوافقية الثنائية

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ملخـص

في هذا البحث، قدمنا مبدأ الانعكاس لصنف المعادلات التفاضلية الجزئية الناقصية ذات المعاملات المتغيرة في حالة الاقترانات ∑ التوافقية الثنائية. إضافة إلى ذلك أوجدنا بعض الخصائص والعلاقات لتلك الاقترانات.

الكلمات الدالة: مبدأ الانعكاس، اقتران متجانس، اقتران ثنائي متجانس، اقتران متعدد متجانس، معادلة تريكومي، معادلة لابلاس، مبدأ الانعكاس، معادلة الحرارة.

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