***Dirasat*,** Pure Sciences, Volume **38**, No. **1**, 2011

Some Geometric and Analytic Properties of Closed Convex Curves in R^2

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ABSTRACT

In this paper we study closed convex curves in R^2. We calculate the area of a convex domain in R^2 using the support function of its boundary which is a closed convex curve. We use Fourier series of the support function to count the extreme points on a curve of constant width in R^2. Finally, we study closed convex curves in R^2 from an analytic point of view as objects of bounded variation.

Keywords: Convex Curves, Support Function, Constant Width, Functions of Bounded Variation.

1. INTRODUCTION

By a convex domain D in the Euclidean plane R^2 we mean a closed bounded subset of R^2 whose boundary ∂D is a convex curve. Equivalently, choosing a tangent for ∂D, the set D lies entirely on one side of the tangent.

Now let C stand for ∂D and assume that the origin o lies inside C. Let α be the perpendicular distance from o to the tangent at a point on C. If θ is the oriented tangent from the positive $x-$axis to the perpendicular ray, then $\alpha=\alpha(\theta)$ is a single valued function with period 2π. The function α is called the support function of C (Eggleston, 1958) (Fillmore, 1969) (Yaglom, 1961).

An important fact is: in terms of the support function α the parameterization of C is given by

$$f(\theta)=(\alpha\cos\theta-\alpha'\sin\theta,\alpha\sin\theta+\alpha'\cos\theta) \qquad (1)$$

The previous formula for f in terms of the support function α is derived using the idea of the envelope of the tangent lines of a smooth closed convex curve in R^2 (Hsiung, 1981) (Struik, 1950). Formula (1) plays a big role in the study of C as in (Al-Banawi, 2004). For the most recent work in the study of convex domains using the support function, see (Al-Banawi, 2008, 2009) .

The curve C in R^2 is of constant width a if

$$\alpha(\theta)+\alpha(\theta+\pi)=a\ ,\ \forall\theta\in[0,2\pi].$$

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In such a situation $f(\theta+\pi)$ is said to be opposite to $f(\theta)$ and tangents at the points $f(\theta), f(\theta+\pi)$ are parallel. Curves of constant width in R^2 were studied in (Mellish, 1931). Mellish's work was explained in (Robertson, 1984).

2. AREA OF CONVEX DOMAINS

Our first result is interesting in the sense that it provides a formula for calculating the area of a convex domain once the support function of its boundary is known.

Theorem 1. Let D be a convex domain with boundary C whose support function is α. Then the area of D is

$$A=\frac{1}{2}\int_0^{2\pi}\alpha(\alpha+\alpha'')d\theta \qquad (2)$$

Proof. We apply Green's Theorem as D is a simply connected domain where

$$A=\frac{1}{2}\oint_C xdy-ydx$$

with

$$x=\alpha\cos\theta-\alpha'\sin\theta$$

and

$$y=\alpha\sin\theta+\alpha'\cos\theta.$$

Observe that $xdy-ydx=\alpha^2+\alpha\alpha''$. Since C has period 2π, formula (2) is proved. ◊

Example 1. Consider the circle $f(\theta)=(r\cos\theta+c_1,r\sin\theta+c_2)$, where (c_1,c_2) is the centre and r is the radius. Using (1), the support function of f is

$$\alpha(\theta)=c_1\cos\theta+c_2\sin\theta+r.$$

Thus,

$$\alpha(\alpha+\alpha'')=r(c_1\cos\theta+c_2\sin\theta+r).$$

Hence the area of a circle is

$$A=\frac{1}{2}\int_0^{2\pi} r(c_1\cos\theta+c_2\sin\theta+r)\,d\theta=\pi r^2 .$$

3. FOURIER SERIES OF THE SUPPORT FUNCTION

Let f be a curve of constant width a in R^2. Since the support function $\alpha(\theta)$ is continuous with a continuous derivative and period 2π, it has a Fourier expansion

$$\alpha(\theta)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos k\theta+b_k\sin k\theta) \quad (3)$$

where, $a_0, a_1,\ldots,b_1,b_2,\ldots$ are constants. The above series converges to $\alpha(\theta)$ for all $\theta\in R.$ Since $\alpha(\theta)+\alpha(\theta+\pi)=a$, then

$$a_0=a,$$
$$a_{2k}=b_{2k}=0,\ k=1,2,\ldots$$

Thus,

$$\alpha(\theta)=\frac{a}{2}+\sum_{k=1}^{\infty}(a_{2k-1}\cos(2k-1)\theta+b_{2k-1}\sin(2k-1)\theta) \quad (4)$$

Now we take the particular case where

$$\alpha(\theta)=\frac{a}{2}+a_{2k-1}\cos(2k-1)\theta+b_{2k-1}\sin(2k-1)\theta \quad (5)$$

which is the essence of the next theorem.

Theorem 2. Let f be a curve of constant width a in R^2 with a support function α as in (5). Then the number of vertices of f is $2n$ where n is odd, $n\ge 3$.

Proof. Recall that $f(\theta_0)$ is a vertex of f if the curvature at $f(\theta_0)$ is extremum.

It is proved in (Al-Banawi, 2008) that the curvature of f is $K=\dfrac{1}{\alpha+\alpha''}$. It is equivalent to work with $\beta=\alpha+\alpha''$. Now

$$\beta(\theta)=\frac{a}{2}+4k(1-k)[a_{2k-1}\cos(2k-1)\theta+ b_{2k-1}\sin(2k-1)\theta].$$

Thus,

$$\beta'(\theta)=4k(k-1)(2k-1)[a_{2k-1}\sin(2k-1)\theta-b_{2k-1}\cos(2k-1)\theta] = 4k(k-1)(2k-1)A\sin((2k-1)\theta-\psi)$$

where

$$\sin\psi=\frac{b_{2k-1}}{A},\ \cos\psi=\frac{a_{2k-1}}{A} \text{ and } A=\sqrt{a^2_{2k-1}+b^2_{2k-1}} .$$

Now β' has $2(2k-1)$ solutions on $[0,2\pi)$. So take $n=2k-1$ which is odd, hence the number of vertices is $2n$. If $k=1$, then α is the support function of a circle. So $k\ge 2$, that is $n\ge 3$ as required. ◊

Example 2. Consider the function $\alpha(\theta)=2+\cos 5\theta,\ \theta\in[0,2\pi],$ corresponding to $k=3.$ Now $\beta'(\theta)=120\sin 5\theta,\ \forall\theta\in[0,2\pi)$, thus the equation $\beta'(\theta)=0$ has 10 solutions. In this case the curvature has maximum value at 5 solutions, namely at $\theta=0,\frac{2\pi}{5},\frac{4\pi}{5},\frac{6\pi}{5},\frac{8\pi}{5}$ and minimum value at 5 solutions, namely at $\theta=\frac{\pi}{5},\frac{3\pi}{5},\pi,\frac{7\pi}{5},\frac{9\pi}{5}$.

4. CONVEX CURVES OF BOUNDED VARIATION

We borrow the next definition from (Apostol, 1979) .

Definition 1. The real-valued function $g:[a,b]\to R$ is said to be of bounded variation on $[a,b]$ if there is a constant $M > 0$ such that $\sum_{i=1}^{n}|g(\theta_i)-g(\theta_{i-1})|\le M$ for all partitions $P=\{\theta_0,\theta_1,\ldots,\theta_n\}$ of $[a,b]$. It should be mentioned that the above definition of bounded variation will be generalized to curves in R^2.

Lemma 1. If $g:[a,b]\to R$ is a monotonically increasing real valued function, then g is of bounded variation.

Proof. For any partition $P=\{\theta_0,\theta_1,\ldots,\theta_n\}$ of $[a,b]$, we have

$$\sum_{i=1}^{n}\left|g(\theta_i)-g(\theta_{i-1})\right|=\sum_{i=1}^{n}(g(\theta_i)-g(\theta_{i-1}))$$
$$=g(\theta_n)-g(\theta_0)$$
$$=g(b)-g(a).\Diamond$$

Theorem 3. Let $f:[a,b]\to R^2$ be a regular curve in R^2. Then the arclength $s(\theta)=\int_a^\theta\|f'(\theta)\|d\theta$ is of bounded variation.

Proof. The regularity of f is equivalent to $f'(\theta)\neq(0,0),\forall\theta\in[a,b]$. Now $s'(\theta)=\|f'(\theta)\|>0$, so $s(\theta)$ is a monotically increasing real valued function of θ.Hence by Lemma 1, $s(\theta)$ is of bounded variation. ◊

Theorem 4. The regular curve $f:[a,b]\to R^2$ is of bounded variation.

Proof. By applying the mean value theorem on the interval $[\theta_{i-1},\theta_i]\subset[a,b]$ and the triangle inequality, we have for some, $\theta\in[\theta_{i-1},\theta_i]$

$$\sum_{i=1}^{n}\|f(\theta_i)-f(\theta_{i-1})\|=\sum_{i=1}^{n}\left\|\int_{\theta_{i-1}}^{\theta_i}f'(\theta)d\theta\right\|$$
$$\leq\sum_{i=1}^{n}\int_{\theta_{i-1}}^{\theta_i}\|f'(\theta)\|d\theta$$
$$=\int_a^b\|f'(\theta)\|d\theta$$
$$=L$$

where L is the length of f. Thus, f is of bounded variation on $[a,b]$.

It should be mentioned here that Lemma 1, Theorems 3 and 4 are well-known in (Apostol, 1979) (Rudin, 1976).

Our final result is about convex curves of bounded variation. ◊

Theorem 5. Let f be a convex curve in R^2. Then the curvature of f admits an integral function of bounded variation.

Proof. Since f is a regular curve in R^2, f can be reparametrized as a function of arclength s. Then the curvature of f is $K=\frac{d\phi}{ds}$ where $\phi(s)$ is the angle between the tangent at $f(s)$ and the positive $x-$ axis (Hsiung, 1981). Since the image of f is convex, $\phi(s)$ defines a monotonically increasing function, hence by Lemma 1, $\phi(s)$ is of bounded variation.

Now

$$\phi(s)=\int K(s)ds$$
$$=\int\frac{1}{\alpha(s)+\alpha''(s)}ds$$

as required. ◊

5. SUMMARY

The support function is a good tool to describe closed convex curves in R^2. The area of a convex domain in R^2 using the support function of its boundary, which is a closed convex curve, has been calculated. Fourier series of the support function has been used to count the extreme points on a curve of constant width in R^2. Finally, the study of closed convex curves in R^2 from an analytic point of view leads to a list of objects of bounded variation.

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بعض الخواص الهندسية والتحليلية للمنحنيات المغلقة المحدبة في المستوى

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ملخص

سندرس في هذا البحث المنحنيات المغلقة المحدبة في R^2 . سنحسب مساحة المجال المحدب في R^2 باستخدام الاقتران الداعم لمحيطه الذي هو منحنى مغلق محدب. سنستخدم متسلسلة فورير للاقتران الداعم كي نعد نقاط القيم القصوى لمنحنى ذي عرض ثابت في R^2 . أخيراً سندرس المنحنيات المغلقة المحدبة في R^2 من وجهة نظر تحليلية كعناصر من ذوات التغير المحدود.

الكلمات الدالة: المنحنيات المغلقة، الاقتران الداعم، عرض ثابت.

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