On Lord Kelvin’s Principle
and Some Properties of \( \Sigma \)-Polyharmonic Functions

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ABSTRACT

In this paper, we extend Lord Kelvin’s principle to \( \Sigma \)-polyharmonic functions for solutions of a class of elliptic partial differential equations with variable coefficients. Relations on these solutions and some properties of \( \Sigma \)-Polyharmonic functions are found too. At the end, we utilize an application.

KEYWORDS: Elliptic partial differential equations, Lord Kelvin’s principle, Harmonic function, \( \Sigma \)-harmonic functions, \( \Sigma \text{-} * \)-harmonic functions, \( \Sigma \)-biharmonic functions, \( \Sigma \)-polyharmonic functions, Generalized Laplace’s equation, Generalized Tricomi’s equation, Heat equation, N: Natural numbers and R: Real numbers.

1. INTRODUCTION

The objective of this paper is to extend Lord Kelvin’s principle to \( \Sigma \)-polyharmonic function of order \( p \), where \( p \) is a positive integer, and also to present relations on solutions of the equation:

\[
L_{b,k}^p [U] = 0,
\]

where the operator \( L_{b,k} \) is given by:

\[
L_{b,k} = \sum_{i=1}^{n} b_i \left( \frac{\partial^2 U}{\partial x_i^2} + k_i \frac{\partial U}{\partial x_i} \right) ; \quad b_i \text{ and } k_i \text{ are real constants for } i = 1,2,\ldots,n.
\]

The domain of the operator \( L_{b,k} \) is the set of all real-valued functions \( U(x_1,x_2,\ldots,x_n) \in C^2(E) \) and \( E \) is the regularity domain of \( U \) in \( \mathbb{R}^n \).

Remark (1): Equation (1) includes some of the following equations as special cases:

1. Generalized Laplace’s equation

\[
L_{0,0}^p [U] = \sum_{i=1}^{n} \frac{\partial^2 U}{\partial x_i^2} = 0, \text{ if } b_i = k_i = 0, i = 1,2,\ldots,n.
\]

2. Generalized Tricomi’s equation

\[
L_{0,1}^p [U] = \sum_{i=1}^{n} \left( \frac{\partial^2 U}{\partial x_i^2} + k_i \frac{\partial U}{\partial x_i} \right) = 0 \text{ if } b_i = 0, i = 1,2,\ldots,n.
\]

Definition: The function \( U \) is called \( \Sigma \)-polyharmonic in a region \( E \) of the \( n \)-dimensional space, if \( U \in C^{2p}(E) \) and satisfies the equation

\[
L_{b,k}^p [U] = 0, \text{ \( \Sigma \)-biharmonic if } U \in C^4(E) \text{ and satisfies the equation } L_{b,k}^2 [U] = 0, \text{ \( \Sigma \)-harmonic if } U \in C^2(E) \text{ and satisfies the equation } L_{b,k} [U] = 0, \text{ and } \Sigma \text{-} * \text{-harmonic if } U \in C^2(E) \text{ and satisfies the equations }
\]

\[
L_{b,k}^* \left[ \sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j} \right]^m U = 0, \quad m \in \mathbb{N}, \text{ and } L_{b,k}^* [U] = 0,
\]

where \( L_{b,k}^* = \sum_{i=1}^{n} b_i x_i^{2q} \left( \frac{\partial^2}{\partial x_i^2} + k_i \frac{\partial}{\partial x_i} \right) ; b_i \neq 0 \text{ for some } i \).

Remark (2): Some results related the \( \Sigma \)-harmonic functions:

1. For \( p,q \in \mathbb{N} \), then

\[
L_{b,k}^p \left[ \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right]^q U = \left( 2p(1-b) + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right) L_{b,k}^p U,
\]

where \( b_i = b, i = 1,2,\ldots,n \). [Rabadi, 1983]

2. For \( b_i \in \mathbb{R} \), then

\[
L_{b,k} \left[ \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right] U = \left( 2 + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \right) L_{b,k} U - 2L_{b,k}^* U,
\]

where \( 0 \neq b_i \neq 1, i = 1,2,\ldots,n \). [Al - Nasiri, 2003]

3. \( \Sigma \)-harmonic gives \( \Sigma \)-harmonic; Take \( b_i = 1, i = 1,2,\ldots,n \) in equation \( L_{b,k}^* [U] = 0 \).
2. Lord Kelvin’s Principle for $\Sigma$-Polyharmonic Functions

We extend Lord Kelvin’s principle to $\Sigma$-polyharmonic function of order $p$, where $p \in \mathbb{N}$. Now, we will investigate two cases:

**Case (1):** If $b_i \neq 1$, for $i = 1,2,...,n$.

**Theorem (1):** Let $U(x_1, x_2, ..., x_n)$ be a $\Sigma^*$-harmonic. If the function $V$ is defined by

$$ V(x_1, x_2, ..., x_n) = r_1^n U(x_1, x_2, ..., x_n), $$

where

$$ r_1^2 = \sum_{i=1}^{n} x_i^{2(1-b_i)} - b_i, $$

then Lord Kelvin’s principle for $\Sigma$-polyharmonic functions in this case has the form:

$$ V(x_1, x_2, ..., x_n) = r_1^{-n-2p+\sum_{i=1}^{n} p_i} U(y_1, y_2, ..., y_n). $$

**Proof:** By direct computation of derivatives we obtain

$$ L_{b,k}[V] = r_1^{m-2} \left( m - 2 + n + \sum_{i=1}^{n} p_i \right) \left( mU - 2 \sum_{j=1}^{n} y_j \frac{\partial U}{\partial y_j} \right) + r_1^{m-4} L_{0,p}[U]. $$

(2)

Where

$$ L_{0,p} = \sum_{j=1}^{n} \left( \frac{\partial^2}{\partial y_j^2} + \frac{p_j}{y_j} \frac{\partial}{\partial y_j} \right). $$

If $U = U(x_1, x_2, ..., x_n)$ is a $\Sigma$-harmonic function, then

$$ m - 2 \sum_{j=1}^{n} y_j \frac{\partial U}{\partial y_j} = 0 $$

is $\Sigma$-harmonic since

$$ L_{b,k}[V] = m - 2 \sum_{j=1}^{n} y_j \frac{\partial U}{\partial y_j}.$$

Now, we apply the operator $L_{b,k}$ on equation (2) to obtain a $\Sigma$-polyharmonic function of order $p$:

$$ L_{b,k}[V] = r_1^{m-4} \left( m - 2 + n + \sum_{i=1}^{n} p_i \right) \left( mU - 2 \sum_{j=1}^{n} y_j \frac{\partial U}{\partial y_j} \right) + r_1^{m-4} L_{0,p}[U]. $$

By part (2) of Remark (2) above the function

$$ \left( m - 2 + n + \sum_{i=1}^{n} p_i \right) \left( mU - 2 \sum_{j=1}^{n} y_j \frac{\partial U}{\partial y_j} \right) \text{ is } \Sigma\text{-harmonic}. $$

Proceeding in this manner, we obtain

$$ L_{b,k}^p[V] = r_1^{m-2p} \left( \prod_{j=1}^{p} \left( m - 2j + n + \sum_{i=1}^{n} p_i \right) \right) \left( \prod_{j=1}^{p} \left( m - 2p - j + \sum_{i=1}^{n} p_i \right) \right) U. $$

(3)

If $V = r_1^m U$ is $\Sigma$-polyharmonic function, then

$$ m - 2j + n + \sum_{i=1}^{n} p_i = 0, \quad j = 1,2,...,p. $$

So,

$$ m = n - 2p + \sum_{i=1}^{n} p_i. $$

Thus, Lord Kelvin’s principle in this case has the form

$$ V(x_1, x_2, ..., x_n) = r_1^{-n-2p+\sum_{i=1}^{n} p_i} U(y_1, y_2, ..., y_n). $$

**Case (2):** If $b_i = 1$ and $k_i = 1$, for $i = 1,2,...,n$.

**Theorem (2):** Let $U(x_1, x_2, ..., x_n)$ be a harmonic function, if the function $W$ is defined by

$$ W(x_1, x_2, ..., x_n) = r_2^n U(z_1, z_2, ..., z_n), $$

where

$$ r_2^2 = \sum_{i=1}^{n} \left( \ln x_i \right)^2, \quad z_i = \frac{\ln x_i}{r_2^2}, \quad i = 1,2,...,n; $$

then Lord Kelvin’s principle for $\Sigma$-polyharmonic functions in this case has the form:
\[ W(x_1, x_2, \ldots, x_n) = r_2^{-(n-2)p} U(z_1, z_2, \ldots, z_n). \]

**Proof:** Clear that
\[
L_{1,1}[W] = r_2^{m-2} (m + n - 2) \left\{ mU - 2 \sum_{j=1}^{n} z_j \frac{\partial U}{\partial z_j} \right\} + 2 \left( m - 2 \sum_{j=1}^{n} z_j \frac{\partial U}{\partial z_j} \right) L_{0,0}[U] = 0,
\]
thus
\[
m - 2 \sum_{j=1}^{n} z_j \frac{\partial U}{\partial z_j} U = \text{harmonic}. \]

Proceeding in this manner, we obtain
\[
L_{1,1}[W] = r_2^{m-4} \left( m + n - 2 \right) \left( m + n + 4 \right) \left\{ m - 2 \sum_{j=1}^{n} z_j \frac{\partial U}{\partial z_j} \right\} + \left( m - 2 \right) \left( m + n + 4 \right) \left\{ m - 2 \sum_{j=1}^{n} z_j \frac{\partial U}{\partial z_j} \right\} U.
\]

Then by part (1) of Remark (2) the function
\[
\left\{ m - 2 \sum_{j=1}^{n} z_j \frac{\partial U}{\partial z_j} \right\} U = \text{harmonic}. \]

Proceeding in this manner, we obtain
\[
L_{1,1}[W] = r_2^{m-2p} \prod_{j=1}^{p} \left( m - 2j + n \right) \left\{ m - 2j + 2 \sum_{j=1}^{n} z_j \frac{\partial U}{\partial z_j} \right\} U.
\]

If \( W = r_2^{m} U \) is \( \Sigma \)-polyharmonic function, then
\[ m - 2j + n = 0, \quad j = 1, 2, \ldots, p. \]

Therefore, \( W(x_1, x_2, \ldots, x_n) = r_2^{-(n-2)p} U(z_1, z_2, \ldots, z_n). \]

To present some relation on solutions of equation \( L_{b, k}[U] = 0 \) by using Lord Kelvin’s principle, we will investigate two cases:

**Case (1):** If \( b_i \neq 1 \), for \( i = 1, 2, \ldots, n \).

**Corollary (1):** Let \( U(x_1, x_2, \ldots, x_n) \) be a \( \Sigma \)-harmonic function. Then the function \( V \) defined by:
\[
V(x_1, x_2, \ldots, x_n) = r_1^{-(n-2)p} U(y_1, y_2, \ldots, y_n), \quad (6)
\]
where
\[
2(1-b_i) \quad x_i = \frac{1-b_i}{x_i}, \quad y_i = \frac{1-b_i}{x_i},
\]
\[
p_i = k_i - b_i, \quad i = 1, 2, \ldots, n,
\]
is a solution of equation \( L_{b, k}[V] = 0 \). In particular, if \( b_i = k_i = 0, i = 1, 2, \ldots, n \), then (6) is a solution of \( L_{b, k}[V] = 0 \). Also, if \( b_i = 0, i = 1, 2, \ldots, n \), then (6) is a solution of \( L_{b, k}[V] = 0 \).

**Proof:** Follows from equation (3).

**Corollary (2):** Let \( U(x_1, x_2, \ldots, x_n) \) be a harmonic function, then the function \( W(x_1, x_2, \ldots, x_n) = r_2^{-(n-2)p} U(z_1, z_2, \ldots, z_n) \), where
\[
r_2^2 = \sum_{i=1}^{n} (\ln x_i)^2, \quad z_i = \frac{\ln x_i}{r_2^2}, \quad i = 1, 2, \ldots, n,
\]
is a solution of equation \( L_{b, k}[W] = 0 \).

**Proof:** Follows from equation (5)

### 3. Some Properties of \( \Sigma \)-Polyharmonic Functions

In this section, we present some generalized properties of the operator \( L_{b, k} \).

**Lemma:** For \( p, q \in \mathbb{N} \), where
\[
L = x_j^{2b_0} \left( \frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right). \]
We have
\[
L_{b, k}^p\left[ l^q[U]\right] = l^q\left[ L_{b, k}^p[U]\right]. \quad (7)
\]

**Proof:** Since
\[
L_{b, k}[U] = \sum_{j=1}^{n} 2b_0 x_j \left( \frac{\partial^2}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial}{\partial x_i} \right) \left[ \frac{2b_0}{x_i} \left( \frac{\partial^2 U}{\partial x_i^2} + \frac{k_i}{x_i} \frac{\partial U}{\partial x_i} \right) \right]
\]
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\[
\begin{align*}
\left[\sum_{j=1}^{n} x_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} \right] & = x_i \frac{\partial}{\partial x_i} \left[ x_i \frac{\partial U}{\partial x_i} \right] + \left[ x_i \frac{\partial^2 U}{\partial x_i \partial x_i} \right] + \sum_{j=1}^{n} x_{ij} \left( \frac{\partial^2 U}{\partial x_i \partial x_j} + \frac{k_j \partial U}{\partial x_j} \right) + x_{ij} \left( \frac{\partial^2 U}{\partial x_i \partial x_j} + \frac{k_i \partial U}{\partial x_i} \right) \\
& = x_i \left( \frac{\partial^2 U}{\partial x_i \partial x_i} + \frac{k_i \partial U}{\partial x_i} \right) + \sum_{j=1}^{n} x_{ij} \left( \frac{\partial^2 U}{\partial x_i \partial x_j} + \frac{k_j \partial U}{\partial x_j} \right) \\
& = x_i \left( \frac{\partial^2 U}{\partial x_i \partial x_i} + \frac{k_i \partial U}{\partial x_i} \right) + \sum_{j=1}^{n} x_{ij} \left( \frac{\partial^2 U}{\partial x_i \partial x_j} + \frac{k_j \partial U}{\partial x_j} \right)
\end{align*}
\]

From (10) and (11), we have

\[
L_{b,k} \left[ x_i \frac{\partial U}{\partial x_i} \right] = x_i \frac{\partial}{\partial x_i} \left[ x_i \frac{\partial U}{\partial x_i} \right] + 2(1-b) \frac{\partial^2 U}{\partial x_i \partial x_i} + 2(1-b) \frac{\partial U}{\partial x_i}
\]

and

\[
L_{b,k} \left[ x_i \frac{\partial L_{b,k} U}{\partial x_i} \right] = x_i \frac{\partial}{\partial x_i} \left[ x_i \frac{\partial L_{b,k} U}{\partial x_i} \right] + 2(1-b) \frac{\partial^2 L_{b,k} U}{\partial x_i \partial x_i} + 2(1-b) \frac{\partial L_{b,k} U}{\partial x_i}
\]

Assume that (8) holds for \( p = \alpha \), then

\[
L_{b,k} \left[ x_i \frac{\partial U}{\partial x_i} \right] = x_i \frac{\partial}{\partial x_i} \left[ x_i \frac{\partial \alpha \frac{\partial^2 L_{b,k} U}{\partial x_i \partial x_i} + 2(1-b) \frac{\partial \alpha \frac{\partial L_{b,k} U}{\partial x_i}}{\partial x_i} \right]
\]

Applying the operator \( L_{b,k} \) to both side of (11), we get

\[
L_{b,k} \left[ x_i \frac{\partial \alpha \frac{\partial^2 U}{\partial x_i \partial x_i} + 2(1-b) \frac{\partial \alpha \frac{\partial U}{\partial x_i}}{\partial x_i} \right] = L_{b,k} \left[ x_i \frac{\partial \alpha \frac{\partial^2 L_{b,k} U}{\partial x_i \partial x_i} + 2(1-b) \frac{\partial \alpha \frac{\partial L_{b,k} U}{\partial x_i}}{\partial x_i} \right]
\]

From (10) and (12), we have
4. Application on the Heat Equation

In this section, we utilize an application on equation $L_{b,k}[U] = 0$, to see this consider the solution

$$W(x_1, x_2, x_3, x_4, x_5, 0, ..., 0) = e^{-x_1^2/4\beta} U(t, x_3, x_4, x_5, 0, ..., 0),$$

$$t = x_1 + ix_2, s = x_1 - ix_2, t^2 = -1, \beta \in R^+.$$  \hspace{1cm} (13)

of equation $L_{b,k}[W] = 0$, we obtain

$$\sum_{j=3}^{5} x_j \left( \frac{\partial^2 U}{\partial x_j^2} + \frac{k_j}{s_j} \frac{\partial U}{\partial x_j} \right) = \frac{1}{\beta} \frac{\partial U}{\partial t}, \beta \in R^+. \hspace{1cm} (15)$$

That is, if $U = U(t, x_3, x_4, x_5, 0, ..., 0)$ is a solution of the heat equation (15), then (13) is a solution of (14).

REFERENCES


