***Dirasat*, Pure Sciences, Volume 34, No. 1, 2007**

On Operator Equation *TA–AT*=C*

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ABSTRACT

Let $\mathcal{H}$ be a separable complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Let A, B, C be elements in $\mathcal{B}(\mathcal{H})$. The equation $AX-XB=C$, which is called Sylvester's equation or Rosenbium's equation has been studied extensively by many mathematicians (see for example Bhatia and Rosenthal, 1997).

More recently, equations of the form $TA-AT^*=C$, where T^* represents the adjoint of T, began to receive more and more attention (Molnár, 1996; Molnár, 1994; Molnár, 1996; Semrl, 1994; Semrl, 1999; Semrl, 1991).

In this paper, we give necessary conditions for equation of the form $TA-AT^*=f(A)$ to have a solution in case A is a normal operator.

Keywords: Normal Operator, The Equation $TA-AT^*=C$

INTRODUCTION

Let $\mathcal{H}$ be a separable complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Let A, B, C be elements in $\mathcal{B}(\mathcal{H})$. The equation $AX-XB=C$, which is called Sylvester's equation or Rosenbium's equation, has been studied extensively by many mathematicians (Bhatia and Rosenthal, 1997).

More recently, equations of the form $TA-AT^*=C$ were studied by L. Molnár and P. Semrl (Molnár, 1996; Molnár, 1994; Molnár, 1996; Semrl, 1994; Semrl, 1999; Semrl, 1991)

In this paper, we give necessary conditions for equation of the form $TA-AT^*=f(A)$ to have a solution in case A is a normal operator.

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Received on 29/5/2005 and Accepted for Publication on 22/11/2006.

1. PRELIMINARIES

Let A be a bounded linear operator on the separable complex Hilbert space $\mathcal{H}$. A is called a Hermitian operator if $A^*=A$ and A is skew-Hermitian if $A=-A^*$, where A^* is the adjoint of A. Recall that the spectrum of A, denoted by $\sigma(A)$, is defined by $\sigma(A)=\{\lambda\in\mathbf{C} : A-\lambda I \text{ is not invertible}\}$. It is known that if A is Hermitian, then $\sigma(A)$ consists of real numbers and if A is skew-Hermitian ,then $\sigma(A)$ consists of pure imaginary numbers, i.e. $\sigma(A)\subseteq i\mathcal{R}$, where $\mathcal{R}$ is the set of real numbers (Berberian, 1976; Halmos, 1957).

The following theorem shows that the equation $TA-AT^*=C$ for fixed A and C may not have a solution.

Theorem 1.1. *Let $A \in \mathcal{B}(\mathcal{H})$ such that $A+A^*$ is not invertible. Then, the equation $TA-AT^*=iI$ has no solution in $\mathcal{B}(\mathcal{H})$.*

Proof. Assume that $TA-AT^*=iI$ for some $T \in \mathcal{B}(\mathcal{H})$. Then, $A^*T^*-TA^*=-iI$. Hence $TA-AT^*-A^*T^*+TA^*=2iI$, therefore

$$T(A+A^*)-(A^*+A)T^*=2iI$$
$$=T(A+A^*)-(T(A+A^*))^*$$

If we put $B=A+A^*$, then it is clear that B is

Hermitian, $TB-(TB)^* = 2iI$, and $TB\neq 0$. This implies that $TB = (iI + (TB)^*) + iI$. If we put $C = iI + (TB)^*$, then $C=TB-iI$. Note that C is Hermitian, in fact,

$$C^* = (iI + (TB)^*)^* = -iI + TB = C.$$

Hence, as stated by Halmos (1957), the spectrum of C, $\sigma(C)$ consists of real numbers. On the other hand, since $A+A^*$ is Hermitian and not invertible, then it can be checked easily that TB is not invertible, and hence $0 \in \sigma(TB)$. But $C=TB-iI$, then by the spectral mapping theorem (Taylor and Lay, 1980), $-i\sigma(C)$; which is a contradiction.

In a similar way we prove the following:

Theorem 1.2. *Let $A \in \mathcal{B}(\mathcal{H})$ such that $A-A^*$ is not invertible. Then, the equation $TA-AT^*=I$ has no solution in $\mathcal{B}(\mathcal{H})$.*

The following proposition shows that if the assumptions of non-invertiblity are dropped in Theorem 1.1 and Theorem 1.2; then the equations may have solutions.

Proposition 1.3. *Let A be an invertible Hermitian operator. Then the equation $TA-AT^*=C$ has a solution $T \in \mathcal{B}(\mathcal{H})$* if and only if C is skew-Hermitian.

Proof. If the equation has a solution T, then, it is easily checked that C is skew-Hermitian. For the converse, if C is skew-Hermitian, and A is invertible, then it is easily seen that $T = \frac{1}{2}CA^{-1}$ is a solution for the equation $TA-AT^*=C$.

In a similar manner, one can prove the following:

Proposition 1.4. *Let A be an invertible skew-Hermitian operator. Then, the equation $TA-AT^*=C$ has a solution in $\mathcal{B}(\mathcal{H})$ if and only if C is Hermitian.*

2. NORMAL OPERATOR

Let $A \in B(H)$, A is called normal if $AA^*=A^*A$, (Berberian, 1976). If $p(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$ is a complex polynomial, where a_i a complex number, $i = 0,1,\cdots,n$, we define

$$p(A) = a_0 I + a_1 A + \cdots + a_n A^n$$

Recall that an element $\lambda \in \sigma(A)$ is called an eigenvalue of A if there exists a non-zero vector $x \in \mathcal{H}$ such that $Ax=\lambda x$. And λ is called an approximate eigenvalue if there exists a sequence $\{u_n\}$ of unit vectors in $\mathcal{H}$ such that $\|(A-\lambda I)u_n\| \to 0$.

Equivalently, for each $\varepsilon>0$ there exists a unit vector $x \in \mathcal{H}$ such that $\|Ax-\lambda x\| \langle \varepsilon$, (Halmos, 1982). It is known that if A is a normal operator, then every element in the spectrum is an approximate eigenvalue, (Halmos, 1957).

In this section, we study the equation $TA-AT^*=p(A)$ for normal operators A. Before we state our main result of this section, we need the following:

Lemma 2.1. *Let T be any element in $\mathcal{B}(\mathcal{H})$, and let p be any polynomial. If λ is any approximate eigenvalue for T, then* $p(\lambda)$ is an approximate eigenvalue for $p(T)$.

Proof. Suppose that p is a polynomial and λ is an approximate eigenvalue for T. Let $\varepsilon >0$. There exists a unit vector $x \in \mathcal{H}$ such that $\|(T-\lambda)x\| \langle \varepsilon$. We show that

$$\|(p(T)-p(\lambda))x\| \langle \varepsilon t ,$$

for some positive t in $\mathcal{R}$, where t does not depend on ε.

Suppose that p is the zero polynomial. Let $t=1$. Then

$$\|(p(T)-p(\lambda))x\| \langle \varepsilon t .$$

Suppose that $T=0$, Then $\lambda=0$. Let $t=1$, Then $(p(T)-p(\lambda))x=a_0I-a_0x$, and thus; $\|(p(T)-p(\lambda))x\| \langle \varepsilon t$.

So we can assume that p is a non-zero polynomial and $T\neq 0$. Note that for each $k\geq 1$,

$$a_k(T^k - \lambda^k I) = a_k(T-\lambda I)(T^{k-1} + \lambda T^{k-2} + \cdots + \lambda^{k-1} I) .$$

Hence

$$\|(p(T)-p(\lambda))x\| = \left\|\sum_{k=1}^{n} a_k(T^k - \lambda^k I)x\right\|$$

$$\leq \sum_{k=1}^{n} \left\| a_k(T^k - \lambda^k I)x\right\|$$

$$\leq \sum_{k=1}^{n} \left\| a_k(T^{k-1} + \lambda T^{k-2} + \cdots + \lambda^{k-1})(T-\lambda I)x\right\|$$

$$\leq \|(T-\lambda I)x\| \sum_{k=1}^{n} \left\| a_k (T^{k-1} + \lambda T^{k-2} + \cdots + \lambda^{k-1}) \right\|.$$

Since $\left\|a_k (T^{k-1} + \lambda T^{k-2} + \cdots + \lambda^{k-1} I)\right\|$ is a constant, say b_k, then;

$$\|(p(T) - p(\lambda))x\| < \varepsilon t, \text{ where } t = \sum_{k=1}^{n} b_k .$$

Now, suppose that A is a normal operator. The following theorem shows that if the equation $TA-AT^*=p(A)$ has a solution and if $\lambda \in \sigma(A)$, $\lambda \neq 0$, then; $\frac{p(\lambda)}{\lambda}$ is a pure imaginary number.

Theorem 2.2. *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator, and let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial. Then*;

(1) *If the equation $TA-AT^*=p(A)$ has a solution T in $\mathcal{B}(\mathcal{H})$, then for each $\lambda \in \sigma(A)$, if $\lambda = 0$ then $p(\lambda)=0$ and otherwise $\frac{p(\lambda)}{\lambda}$ is a pure imaginary number, i.e, an element in $i\mathcal{R}$.*

(2) *If the equation $TA^*-A^*T^*=p(A)$ has a solution T in $\mathcal{B}(\mathcal{H})$, then for each $\lambda \in \sigma(A)$, $\lambda p(\lambda)$ is a pure imaginary number.*

Proof.

(1) Let $\varepsilon>0$ and let $\lambda \in \sigma(A)$, Suppose that $T=0$, Then, $p(A)=TA-AT^*=0$. By the Spectral Mapping Theorem $p(\sigma(A))=\sigma(p(A))=\sigma(0)=\{0\}$. It follows that $p(\lambda)=0$. Hence, if $\lambda=0$, then $p(\lambda)=0$, and otherwise $\frac{p(\lambda)}{\lambda}=0=i0$ belongs to $i\mathcal{R}$. Suppose that $T \neq 0$. Since λ is an approximate eigenvalue for A, then there exists a unit vector x in H such that;

(2.1) $\qquad ||Ax-\lambda x|| \, ||T^*|| < \varepsilon, ||x||=1.$

Hence by lemma 2.1 and its proof, $p(\lambda)$ is an approximate eigenvalue for $p(A)$. Thus; $||p(A)x-p(\lambda)x|| \, ||x|| < \varepsilon t$, $t>0$, $||x||=1$, (see the proof of Lemma 2.1). By Schwarz inequality, $|\langle p(A)x-p(\lambda)x,x\rangle| < \varepsilon t$ and hence

(2.2) $\qquad |\langle p(A)x,x\rangle - p(\lambda)| < \varepsilon t.$

On the other hand, $p(A)-(TA-AT^*)=0$ implies $\langle p(A)-(TA-AT^*)x,x\rangle = 0.$

and hence

(2.3) $\qquad |\langle p(A)x,x\rangle - \langle Ax,T^*x\rangle + \langle T^*x,A^*x\rangle| = 0.$

Now, the normality of A implies $||Ay||=||A^*y||$ for all $y \in \mathcal{H}$. Hence from equation (2.1) we get

(2.4) $\qquad \|T^*\| \|A^*x - \bar{\lambda}x\| < \varepsilon.$

From equations (2.1),(2.4) we deduce, using Schwarz inequality, that

(2.5) $\qquad |\langle Ax,T^*x\rangle - \lambda\langle x,T^*x\rangle| < \varepsilon.$

and

(2.6) $\qquad |\langle T^*x,A^*x\rangle + \lambda\langle T^*x,x\rangle| < \varepsilon$

By adding equations (2.2), (2.3), (2.5), (2.6) we get $|p(\lambda)-\lambda\langle (T-T^*)x,x\rangle| < \varepsilon(t+2).$

Since $T-T^*$ is skew-Hermitian, then it is easily seen that $|\langle (T-T^*)x,x\rangle|$ is pure imaginary number, say c. Thus; $|p(\lambda)-\lambda c| < \varepsilon(t+2), \quad c \in i\mathcal{R}.$

Suppose that $\lambda=0$, Then $|p(\lambda)|=|p(\lambda)-\lambda c| < \varepsilon(t+2)$. We have $|p(\lambda)| < \varepsilon(t+2)$ for each $\varepsilon>0$, It follows that $p(\lambda)=0$.

On the other hand, suppose that $\lambda \neq 0$, Then

$$d\left(\frac{p(\lambda)}{\lambda}, i\mathcal{R}\right) = \inf\left(\left|\frac{p(\lambda)}{\lambda} - z\right| : z \in i\mathcal{R}\right) \leq$$

$$\left|\frac{p(\lambda)}{\lambda} - c\right| = \frac{1}{|\lambda|}|p(\lambda) - \lambda c| \langle \frac{\varepsilon(t+2)}{\lambda}.$$ It follows that

$$d\left(\frac{p(\lambda)}{\lambda}, i\mathcal{R}\right) < \frac{\varepsilon(t+2)}{\lambda},$$ for each $\varepsilon>0$. Hence

$$d\left(\frac{p(\lambda)}{\lambda}, i\mathcal{R}\right) = 0,$$ and thus $\frac{p(\lambda)}{\lambda} \in i\mathcal{R}$.

(2) Assume $p(A)=TA^*-A^*T^*$ for some $T \in \mathcal{B}(\mathcal{H})$. As in Part (1), we can assume that $T \neq 0$.

$||Ax-\lambda x|| \, ||T^*|| < \varepsilon, \quad ||x||=1$

$|p(A)x-p(\lambda)x| < \varepsilon t$

Thus, the equations (2.2), (2.3), (2.5), (2.6) become

$|\langle p(A)x,x\rangle - p(\lambda)| < \varepsilon t$

$|\langle p(A)x,x\rangle - \langle A^*x,T^*x\rangle + \langle T^*x,Ax\rangle| = 0,$

$|\langle T^*x,Ax\rangle - \bar{\lambda}\langle T^*x,x\rangle| < \varepsilon,$

$|\langle A^*x,T^*x\rangle - \bar{\lambda}\langle x,T^*x\rangle| < \varepsilon.$

By adding the last four equations we get

$|p(\lambda)-\bar{\lambda}\langle(T-T^*)x,x\rangle| < \varepsilon(t+2)$.

We have $d(\lambda p(\lambda),i\mathcal{R})=\inf\{|\lambda p(\lambda)-z|: z\in i\mathcal{R}\}\le\left|\lambda p(\lambda)-|\lambda|^2 c\right|=|\lambda|\left|p(\lambda)-\bar{\lambda}c\right|\le|\lambda|\,\varepsilon\quad(t+2)$. It follows that $d(\lambda p(\lambda),i\mathcal{R}) \le |\lambda|\,\varepsilon(t+2)$, for each $\varepsilon>0$. Hence, $d(\lambda p(\lambda),i\mathcal{R})=0$, and, $\lambda p(\lambda)\in i\mathcal{R}$.

Because of the importance of the special case $p(\lambda)=\lambda^n$, we record it as a corollary.

Corollary 2.3. *Let* $A \in \mathcal{B}(\mathcal{H})$ *be a non-zero normal operator and the equation* $TA-AT^* = A^n$ *has a solution* $T \in \mathcal{B}(\mathcal{H})$, *when n is a positive integer. Then*,

(1) *If* $n=1$, *then* $A=0$, *i.e. the equation* $TA-AT^*=A$ *has no solution.*

(2) *If* $n\ge 2$, *then* $(\sigma(A))^{n-1}\subseteq i\mathcal{R}$.

Remark 2.4. If A is a normal operator, then so is A^n for each positive integer n. Hence, if the equation $TA-AT^*=A^n$ has a solution in $\mathcal{B}(\mathcal{H})$, then $\sigma(A^{n-1})\subseteq i\mathcal{R}$. Hence A^{n-1} is skew-Hermitian operator (Halmos, 1957), In particular, if $TA-AT^*=A^2$ and A is normal, then A is skew-Hermitian. And if A^n is not skew-Hermitian, the equation $TA-AT^*=A^{n+1}$ has no solution in $\mathcal{B}(\mathcal{H})$.

On the other hand, if A^{n-1} is skew-Hermitian, then $T=\frac{1}{2}A^{n-1}$ is a solution of the equation $TA-AT^*=A^n$.

Example 2.5. Let

$$\mathcal{H}=\left\{x=(\cdots,x_{-1},x_0,x_1\cdots):\|x\|^2=\sum_{i=-\infty}^{\infty}|x_i|^2<\infty,\ x_i\in C\right\}.$$

Let A: $\mathcal{H}\to\mathcal{H}$ be the shift operator defined on the basis $\{e_n\}$ by $Ae_n=e_{n+1}$, $-\infty< n <\infty$. It is easily checked that $A^*e_n=e_{n-1}$. Thus, $AA^*=A^*A$ and A is normal. Note that A is not skew-Hermitian for all $n\ge 1$.

Corollary 2.6. *Let A be an operator in* $\mathcal{B}(\mathcal{H})$ *such that* $A+A^*$ *is not invertible. Let* $p(\lambda)$ *by any polynomial such that* $p(A)-(p(A))^*= cI$, $c\neq 0$. *Then, the equation* $TA-AT^*= p(A)$ *has no solution in* $\mathcal{B}(\mathcal{H})$.

Proof. Assume that the equation $TA-AT^*=p(A)$ has a solution T in $\mathcal{B}(\mathcal{H})$. Then, $A^*T^*-TA^*=(p(A))^*$. Thus, $T(A+A^*)-(A+A^*)T^*=p(A)-(p(A))^*=cI=q(A)$. Note that $A+A^*$ is Hermitian, hence normal and $0\in\sigma(A+A^*)$. Thus, by 2.2 $q(0)=0$ which is a contradiction.

Note. Corollary 2.6 gives another proof for 1.1.

Before we state our next result, let us recall the following from the complex analysis (Nahari, 1968).

Lemma 2.7. *Let D be a bounded simply connected domain in the complex plane* **C**, *with* $\partial D = C$ *is a simple closed contour. Let f be an analytic function in* $D\cup C$. *If* $Re f=0$ *on C, then f is a constant function.*

Theorem 2.8. *Let A be a normal operator in* $\mathcal{B}(\mathcal{H})$, *and let p be a polynomial. Suppose that* $\sigma(A)$ *contains a simple closed contour C.*

(1) *Furthermore, assume that* $\frac{p(\lambda)}{\lambda}$ *is analytic at 0, if 0 belongs to the simple closed contour C union its interior. The equation* $TA-AT^*=p(A)$ *has a solution T in* $\mathcal{B}(\mathcal{H})$ *if and only if there exists c in* $i\mathcal{R}$ *such that* $p(\lambda)=\lambda c$ *for all* λ *in* C.

(2) *The equation* $TA^*-A^*T^*=P(A)$ *has a solution T in* $\mathcal{B}(\mathcal{H})$ *if and only if* $p(\lambda)=0$.

Proof. (1) By theorem (2.2,i), for each $\lambda\in\sigma(A)$, $\frac{p(\lambda)}{\lambda}=c$, $c\in i\mathcal{R}$, and c depends on λ. Since p is analytic in the whole complex plane, it is analytic on $\sigma(A)$. Moreover, by assumption $\frac{p(\lambda)}{\lambda}$ is analytic on C and its interior. Thus, by Lemma 2.7, $\frac{p(\lambda)}{\lambda}$ is a constant function, say $c_0\in i\mathcal{R}$, on the contour C union its interior, and hence $p(\lambda)=c_0\lambda$ for each λ *in* C and $p(A)=c_0A$.

(2) Suppose that $TA^*-A^*T^*=p(A)$ *for some* $T\in\mathcal{B}(\mathcal{H})$. Then, for each $\lambda\in\sigma(A)$, $\lambda p(\lambda)=c$, and $c\in i\mathcal{R}$ may depend on λ. Since $\lambda p(\lambda)$ is analytic on $\sigma(A)$ and $\sigma(A)$ contains a closed contour, then by 2.7 $\lambda p(\lambda)$ is a constant function say c_0, $c_0\in i\mathcal{R}$. Obviously, this can happen only if $p(\lambda)=0$.

The next example shows that theorem 2.8 is false in case $\frac{p(\lambda)}{\lambda}$ is not analytic in the domain bounded by C. Note that $\frac{p(\lambda)}{\lambda}$ is analytic at $\lambda=0$ if and only if $p(0)=0$.

Example 2.9. Let A be the bilateral shift operator, defined in example 2.5. A is unitary i.e. $AA^*=A^*A=I$, hence normal. Recall that the spectrum of A is the boundary of the unit disk D (Halmos, 1982). If $p(\lambda)=1-\lambda^2$, then $TA-AT^*=I-A^*$ where $T=A^*$. Note that $0\in D$ and $\frac{1-\lambda^2}{\lambda}$ is not analytic at 0.

3. ANALYTIC OPERATORS

Let f be a complex analytic function defined on the ball $B_r=\{z\in\mathbf{C}:|z|<r\}$ where $r>0$. Let $A\in\mathcal{B}(\mathcal{H})$ such that $||A||<r$. By Taylor theorem $f(z)=\sum_{n=0}^{\infty}a_n z^n$, and this series converges in $|z|<r$. It is known that $\sum_{n=0}^{\infty}a_n z^n$ converges in $\mathcal{B}(\mathcal{H})$, (Rajavi and Rosenthal, 1973), and one can define

$$f(A)=\sum_{n=0}^{\infty}a_n A^n .$$

Since $|\sigma(A)|\le ||A||$, where $|\sigma(A)|$ denotes the spectral radius of A, then $\sigma(A)\subset B_r$. In particular, if λ is an eigenvalue for A, then $\sum_{n=0}^{\infty}a_n\lambda^n$ that converges to $f(\lambda)$ is defined. The operator $f(A)$ is called an analytic operator.

Before we give one of our main results in the paper, we need the following:

Lemma 3.1. *Let f and A be as above, and let λ be an eigenvalue for A, then f(λ) is an eigenvalue for f(A). And if Ax=λx, then f(A)x=f(λ)x.*

Proof. For each non-negative integer n, let $p_n(\lambda)=\sum_{k=0}^{n}a_k\lambda^k$. It is easily checked that $p_n(A)x=p_n(\lambda)x$. But $p_n(A)\to f(A)$ and $p_n(\lambda)\to f(\lambda)$, hence $p_n(A)x\to f(A)x$ and $p_n(\lambda)x\to f(\lambda)x$, and thus $f(A)x=f(\lambda)x$.

The next theorem gives necessary conditions for the equation $TA-AT^*=f(A)$ to have a solution.

Theorem 3.2. *Let $A\in\mathcal{B}(\mathcal{H})$ be a normal operator, and let $f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ be analytic function in the ball B_r such that $||A||<r$.*

(1) *If the equation $TA-AT^*=f(A)$ has a solution in $\mathcal{B}(\mathcal{H})$, then for each eigenvalue λ of A, if λ=0, f(λ)=0 and otherwise $\frac{f(\lambda)}{\lambda}$.*

(2) *If the equation $TA^*-A^*T^*=f(A)$ has a solution in $\mathcal{B}(\mathcal{H})$, then, for each eigenvalue λ of A, λf(λ) is a pure imaginary number.*

Proof. (1) Let λ be an eigenvalue for A, and let x be the corresponding eigenvector. Thus, $Ax=\lambda x$, we may assume $||x||=1$. By 3.1 $f(A)x=f(\lambda)x$, hence $\langle f(A)x,x\rangle-f(\lambda)x=0$. Moreover, $f(A)-(TA-AT^*)=0$, hence $\langle f(A)x,x\rangle-\langle Ax,T^*x\rangle+\langle T^*x,A^*x\rangle=0$. Since A is normal $A^*x=\bar{\lambda}x$, hence $\langle Ax-\lambda x,T^*x\rangle=0$ and $\langle T^*x,A^*x-\bar{\lambda}x,\rangle=0$.

It follows now from these equations that

$$f(\lambda)-\lambda\left(\langle x,T^*x\rangle-\langle T^*x,x\rangle\right)=0 .$$

Hence, $f(\lambda)-\lambda\left(\langle (T-T^*)x,x\rangle\right)=0$, and thus $f(\lambda)=\lambda c$, where $c=\langle (T-T^*)x,x\rangle\in i\mathcal{R}$.

(2) We leave the proof of (2) to the reader.

The proof of the following theorem is similar to the proof 2.8, hence it is omitted.

Theorem 3.3. Let A be a normal operator on $\mathcal{H}$. Suppose that $f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ is analytic in some ball B_r such that $||A||<r$, and suppose that the set $\pi_0(A)$, the set of eigenvalues of A, contains a simple closed contour C.

(1) Furthermore, assume that $\frac{f(\lambda)}{\lambda}$ is analytic at 0 if 0 belongs to the simple closed contour C union its interior. Then the equation $TA-AT^*=f(A)$ has a solution in $\mathcal{B}(\mathcal{H})$ if and only if there exists c in $i\mathcal{R}$ such that $f(\lambda)=c\lambda$ for all λ in B_r.

(2) The equation $TA^*-A^*T^*=f(A)$ has a solution T in $\mathcal{B}(\mathcal{H})$ if and only if $f\equiv 0$ on B_r.

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حول المعادلة *TA-AT*=C*

عادل غسان نعوم وأمينة القباني *

ملخـص

ليكن $\mathcal{H}$ فضاء هلبرت قابلاً للفصل و$\mathcal{B}(\mathcal{H})$ جبر المؤثرات الخطية المقيدة على $\mathcal{H}$. لتكن A,B,C عناصر في $\mathcal{B}(\mathcal{H})$. المعادلة التي بالصيغة AX–XB=C والتي تدعى معادلة سلفستر درست باهتمام من قبل العديد من الباحثين.

في الآونة الأخيرة ظهرت دراسة نوع آخر من الدوال التي بدأت تحظى باهتمام أكثر فأكثر، وهي على الصيغة $TA-AT^*=C$، حيث T^* هو المؤثر المرافق الى T.

في هذا البحث ندرس شروطا ضرورية ليكون للمعادلة $TA-AT^*=f(A)$ حل، حيث A و T تنتميان الى $\mathcal{B}(\mathcal{H})$ والدالة f دالة تحليلية.

من أهم النتائج في هذا البحث انه اذا كان المؤثر A سويا وطيفه يحقق مواصفات معينة فإن المعادلة $TA-AT^*=f(A)$ لها حل في جبر المؤثرات الخطية المقيدة $\mathcal{B}(\mathcal{H})$.

الكلمات الدالـة: المعامل القياسي (المعياري)، المعادلة TA-AT*=C.

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تاريخ استلام البحث 2005/5/29، وتاريخ قبوله 2006/11/22.