

Hamilton-Jacobi Formulation of Systems with Gauge Conditions

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ABSTRACT

After reviewing the Hamilton-Jacobi formulation approach of constrained systems using the canonical method, we apply this approach to a system of first-class constraints in the framework of the Dirac method. In constrained systems possessing first-class constraints, some gauge conditions should be applied to convert them into second-class constraints. These conditions reduce the total Hamiltonian in Dirac's method to the canonical one. We show that the Hamilton-Jacobi function in reduced phase space can be obtained in the same manner as for the canonical method. The equations of motion and quantization using the WKB approximation are clarified.

Keywords: Hamilton-Jacobi, Constrained Systems, Gauge Conditions.

1. INTRODUCTION

The study of constrained systems was initiated by Dirac (Dirac, 1950; Dirac, 1964) for the purpose of quantization. He showed that the first step toward quantizing constrained systems was to derive the Hamiltonian from the Lagrangian, treating the coordinates (q's) and momenta (p's) as operators satisfying the commutation relations. He divided the constraints into two types: first-class constraints which have zero Poisson brackets with all other constraints, and second-class constraints which have non-zero Poisson brackets. In the case of second-class constraints, Dirac introduced a new Poisson bracket, the Dirac bracket, which correspond to the commutator. However, for first-class constraints, Dirac imposed certain supplementary conditions on the wave functions.

Following Dirac, it was shown that gauge fixing conditions should be imposed for first-class constraints so as to convert them into second-class constraints

(Faddeev, 1970; Hanson and Regge, 1976; Evans, 1991), $\chi_\alpha = 0$, which are supposed to have a vanishing Poisson bracket with the canonical Hamiltonian H_0 .

Recently, another approach based on the canonical method has been developed to investigate constrained systems (Guler, 1992; Guler, 1992; Rabei and Guler, 1992). The starting point of this method is the variational principle. The Hamiltonian treatment of the constrained systems leads to obtain the equations of motion as total differential equations in many variables, which require the investigation of integrability conditions. The equations of motion are integrable if the corresponding system is a Jacobi system. In this case, one can construct valid canonical phase space coordinates.

The link between the two approaches is studied (Rabei, 1990). It is shown that the Hamilton-Jacobi approach is always in exact agreement with the Dirac approach. However, the Hamilton-Jacobi formalism has been proven to be a good tool for constrained systems (Guler, 1992; Guler, 1992; Rabei and Guler, 1992; Rabei, 1996). In this approach, there is no need to distinguish between the two types of constraints, the first and second-class. For example, the canonical path integral

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representation of constrained systems has been successfully obtained using this formalism (Muslih, 2000; Muslih, 2000).

On the other hand, an interesting model for solving mechanical problems of singular systems of first order Lagrangians has been proposed in the same manner as for regular systems (Rabei et al., 2002). The Hamilton-Jacobi Function (HJF) in configuration space has been obtained by solving the set of Hamilton-Jacobi Partial Differential Equations (HJPDEs) for these singular systems. In addition, The Hamilton-Jacobi theory provides a bridge between classical and quantum mechanics. The principal interest in this theory was based on the hope that it might provide some guidance, concerning the form of a Schrödinger-type quantum theory for constrained fields. Thus, calculating the HJF enables us to quantize constrained systems of first order Lagrangians using the WKB approximation. Depending on this approach, the treatment of higher-order Lagrangians systems has been developed in the same manner, so as to obtain the equations of motion and to quantize constrained systems using the WKB approximation (Hassan et al., 2004). This new approach, due to its very recent development, has been applied to very few examples (Rabei et al., 2002; Rabei et al., 2003; Nawafleh et al., 2004; Nawafleh et al., 2005). In this work we hope to study the construction of the HJF for first-class constraints with gauge conditions in the framework of the Dirac method. In fact, this work is a continuation of previous papers (Rabei et al., 2002; Rabei et al., 2003; Nawafleh et al., 2004; Nawafleh et al., 2005), where we have obtained the HJF and the WKB approximation for singular systems.

The plan of the paper is as follows:

In Section 2, the canonical method for determining the Hamilton-Jacobi function is reviewed briefly. In Section 3, we discuss the formalism of the Dirac method

for treating systems of first-class constraints in the reduced phase space after applying the gauge fixing conditions. In Section 4, an illustrative example covering a system of first-class constraints is introduced. Through this example, the equations of motion and the quantization using the WKB approximation will be clarified. Finally, Section 5 contains some concluding remarks.

2. CANONICAL METHOD

The starting point is the singular Lagrangian $L = L(q_i, \dot{q}_i)$, $i = 1, 2, \dots, N$ with the Hessian matrix

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

of rank $N-R$, $R < N$. The generalized momenta p_i , corresponding to the generalized coordinates q_i , are defined as:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (1)$$

Since the rank of the Hessian matrix is $N-R$, the definition (1) produces R functionally independent relations of the form (Guler, 1992; Guler, 1992)

$$H'_\mu(q_i, p_i) = p_\mu + H_\mu = 0; \quad \mu = 1, \dots, R, \quad (2)$$

which are called primary constraints (Dirac, 1950; Dirac, 1964).

Next, the canonical Hamiltonian H_0 is defined as

$$H_0 = p_a \dot{q}_a + p_\mu \dot{q}_\mu - L; \quad a = 1, 2, \dots, N-R. \quad (3)$$

Here and throughout the paper, Einstein's summation rule for repeated indices is used.

The canonical formulation (Guler, 1992; Guler, 1992; Rabei and Guler, 1992) gives the set of HJPDEs, as

$$\begin{aligned} H'_0 = p_0 + H_0 = \frac{\partial S}{\partial t} + H_0 \left(q_\beta, q_a, p_a = \frac{\partial S}{\partial q_a} \right) &= 0; \\ H'_\mu = p_\mu + H_\mu = \frac{\partial S}{\partial q_\mu} + H_\mu \left(q_\beta, q_a, p_a = \frac{\partial S}{\partial q_a} \right) &= 0. \end{aligned} \quad (4)$$

The equations of motion are obtained as total differential equations in many variables as follows:

$$\begin{aligned} dq_a &= \frac{\partial H'_a}{\partial p_a} dq_a = \frac{\partial H'_0}{\partial p_a} dt + \frac{\partial H'_\mu}{\partial p_a} dq_\mu; \\ dp_i &= -\frac{\partial H'_a}{\partial q_i} dq_a = -\frac{\partial H'_0}{\partial q_i} dt - \frac{\partial H'_\mu}{\partial q_i} dq_\mu. \end{aligned} \quad (5)$$

These equations are integrable (Rabei, 1996) if and only if,

$$dH'_\mu = 0 \quad (6)$$

is identically satisfied or leads to a new secondary constraint. Then, one can solve Eqn.(5) to obtain the coordinates q_a and momenta p_i as functions of q_μ and t .

A general approach for solving the set of HJPDEs for the constrained systems (4) has been studied (Rabei et al., 2002). The general solution is given in the form:

$$S(q_a, q_\mu, t) = f(t) + W_a(E_a, q_a) + f_\mu(q_\mu) + A, \quad (7)$$

where E_a are the $(N-R)$ constants of integration and A is some other constant. The equations of motion can be obtained using the canonical transformations as follows:

$$\lambda_a = \frac{\partial S}{\partial E_a}; \quad p_i = \frac{\partial S}{\partial q_i}, \quad (8)$$

where λ_a are constants and can be determined from the initial conditions. The number of λ_a is equal to the rank of the Hessian matrix.

Equation (8) can be solved to furnish q_a and p_i as follows:

$$q_a = q_a(\lambda_a, E_a, q_\mu, t), \quad p_i = p_i(\lambda_a, E_a, q_\mu, t). \quad (9)$$

The Hamilton-Jacobi function is used to quantize constrained systems using the WKB approximation. The wave function has been obtained as follows:

$$\psi(q_a, q_\mu; t) = \left[\prod_{a=1}^{N-R} \psi_{0a}(q_a) \right] e^{iS(q_a, q_\mu, t)/\hbar}, \quad (10)$$

where

$$\psi_{0a}(q_a) = \frac{1}{\sqrt{p_a}}. \quad (11)$$

This wave function satisfies the conditions:

$$\hat{H}'_0 \psi = 0; \quad \hat{H}'_\mu \psi = 0. \quad (12)$$

These conditions are obtained when the dynamical coordinates and momenta are turned into their corresponding operators:

$$\begin{aligned} q_i &\rightarrow \hat{q}_i = q_i; \\ p_i &\rightarrow \hat{p}_i = \frac{\hbar}{i} \frac{\partial}{\partial q_i}; \\ p_o &\rightarrow \hat{p}_o = \frac{\hbar}{i} \frac{\partial}{\partial t}. \end{aligned} \quad (13)$$

3. DIRAC'S METHOD

In Dirac's method, the primary constraints (2) multiplied by unknown coefficients v_μ 's may be added to the canonical Hamiltonian H_0 to obtain the total Hamiltonian H_T

$$H_T = H_0 + v_\mu H'_\mu. \quad (14)$$

The consistency conditions imply that the time derivative of the primary constraints should be zero

$$\dot{H}'_\mu = \{H'_\mu, H_0\} + v_\nu \{H'_\mu, H'_\nu\} \approx 0, \quad (15)$$

where Dirac's symbol for weak equality has been used. If these equations are consistent, three cases are possible: an equation can give an identity; it can give a linear equation for the v_μ ; or it can give an equation containing only p's and q's, in which case it must be considered as another constraint. The constraints that arise from this procedure will be called secondary.

Primary and secondary constraints are divided into two types: first- class constraints, which have vanishing Poisson brackets with all other constraints; and second-class constraints, which do not have this property. As there is an even number of second-class constraints, these can be used to eliminate the conjugate pairs of p's and q's from the theory by expressing them as functions of the remaining p's and q's. The Dirac Hamiltonian for the

remaining variables is then the canonical Hamiltonian plus the first-class constraints. So, the total Hamiltonian reduces to be

$$H_T(q_i, p_i) = H_0(q_i, p_a) + v_\alpha H'_\alpha(q_i, p_\mu), \quad (16)$$

where H'_α are all the independent remaining first-class constraints, and v_μ 's are arbitrary functions.

This arbitrariness in v_μ 's implies that not all q's and p's are physical variables. To remove this arbitrariness, one has to impose an external gauge constraint for each first-class constraint. Such a gauge fixing, $\chi_\alpha = 0$, is a set of constraints independent of H'_α . Such a choice converts the whole set of constraints $\{H'_\alpha, \chi_\alpha\}$ into second-class constraints. In this case, the total Hamiltonian becomes free of constraints

$$H_T(q_i, p_i) = H_0^*(q_a, p_a). \quad (17)$$

In the same manner as for the canonical method, if we denote H_T by $-p_0$, then Eq.(17) may be written as:

$$p_0 + H_0^*(q_a, p_a) = 0. \quad (18)$$

The corresponding Hamilton-Jacobi equation is:

$$\frac{\partial S}{\partial t} + H_0^*\left(q_a, p_a = \frac{\partial S}{\partial q_a}\right) = 0, \quad (19)$$

which is equivalent to the first equation in set (4).

Thus, we have the following solution:

$$S(q_a, t) = f(t) + W_a(E_a, q_a) + A. \quad (20)$$

Once we have found the Hamilton-Jacobi function, the equations of motion and the wave function of the system can be obtained.

4. EXAMPLE

Consider the singular Lagrangian (Christ-Lee model)

(Christ and Lee, 1980):

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \lambda(x\dot{y} - y\dot{x}) + \frac{1}{2}\lambda^2(x^2 + y^2) - V(x^2 + y^2). \quad (21)$$

The generalized momenta read:

$$\begin{aligned} p_x &= \dot{x} + \lambda y; \\ p_y &= \dot{y} - \lambda x; \\ p_\lambda &= 0, \end{aligned} \quad (22)$$

since the Hessian matrix is of rank two, we have one primary constraint:

$$\phi_1 = p_\lambda \approx 0. \quad (23)$$

The Hamiltonians H_0 and H_T are calculated as:

$$H_0 = \frac{1}{2}(p_x^2 + p_y^2) + \lambda(xp_y - yp_x) + V(x^2 + y^2); \quad (24)$$

$$H_T = H_0 + v_1\phi_1. \quad (25)$$

Invoking the consistency condition (15) for the primary constraint ϕ_1 , we obtain the secondary constraint:

$$\phi_2 = xp_y - yp_x \approx 0. \quad (26)$$

Applying the consistency condition for ϕ_2 , we deduce that no further constraints arise.

The constraints ϕ_1 and ϕ_2 are first-class; the following subsidiary conditions

$$\chi_1 = \lambda; \quad \chi_2 = x - y, \quad (27)$$

can be used as gauge fixing conditions, because the four constraints are now second-class.

When the constraints and gauge fixing conditions are used, the total Hamiltonian reduces to:

$$H_T = H_0^* = \frac{1}{2}(p_x'^2) + V(x'^2), \quad (28)$$

where

$$p_x' = \sqrt{2}p_x; \quad x' = \sqrt{2}x. \quad (29)$$

The corresponding HJ equation is:

$$p_0 + H_0^* = \frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x'}\right)^2 + V(x'^2) = 0. \quad (30)$$

The Hamilton-Jacobi function can then be determined as:

$$S(x', t) = f(t) + W(x', E) + A. \quad (31)$$

Since the reduced Hamiltonian (28) is time independent, one can write $f(t) = -Et$.

Substituting Eq.(31) in Eqn.(30), we get:

$$-E + \frac{1}{2} \left(\frac{\partial W}{\partial x'} \right)^2 + V(x'^2) = 0, \quad (32)$$

which has the following solution:

$$W(x', E) = \int \sqrt{2(E - V(x'^2))} dx'. \quad (33)$$

With these results, the HJF becomes:

$$S = -Et + \int \sqrt{2(E - V(x'^2))} dx' + A. \quad (34)$$

Making use of Eq.(8), the equations of motion arising from the above HJ function can be determined as follows:

$$\lambda = \frac{\partial S}{\partial E} = -t + \int \frac{dx'}{\sqrt{2(E - V(x'^2))}}. \quad (35)$$

For simplicity, let us take $V(x'^2) = x'^2$, then this equation can readily be integrated to give

$$x' = \sqrt{E} \sin[\sqrt{2}(\lambda + t)]. \quad (36)$$

Further,

$$p'_x = \frac{\partial S}{\partial x'} = \sqrt{2E} \cos(\lambda + t). \quad (37)$$

Substituting for x' , we get:

$$p'_x = \frac{\partial S}{\partial x'} = \sqrt{2E} \cos(\lambda + t). \quad (38)$$

We are now in a position to quantize our system. The wave function for this example can be determined as:

$$\psi(x', t) = \psi_0(x') e^{iS/\hbar}, \quad (39)$$

where

$$\psi_0(x') = \frac{1}{\sqrt{p'_x}} = [2(E - x'^2)]^{-\frac{1}{4}}. \quad (40)$$

Now applying the quantum HJ equation (28) to the wave function

$$\hat{H}'_0 \psi = \left[\frac{\hbar}{i} \frac{\partial}{\partial t} - \frac{\hbar^2}{2} \left(\frac{\partial^2}{\partial x'^2} \right) + x'^2 \right] \psi. \quad (41)$$

Then, we have

$$\begin{aligned} \frac{\hbar}{i} \frac{\partial}{\partial t} \psi &= -E\psi; \\ -\frac{\hbar^2}{2} \frac{\partial^2}{\partial x'^2} \psi &= \left[\frac{-\hbar^2}{4(E - x'^2)} - \frac{-5\hbar^2}{2[(E - x'^2)]^2} + (E - x'^2) \right] \psi. \end{aligned} \quad (42)$$

Substituting these back in Eq.(41), we obtain

$$H'_0 \psi = \left[-E + \frac{-\hbar^2}{4(E - x'^2)} - \frac{-5\hbar^2}{2[(E - x'^2)]^2} + (E - x'^2) + x'^2 \right] \psi. \quad (43)$$

In the semi-classical limit $\hbar \rightarrow 0$, we get:

$$H'_0 \psi \equiv 0. \quad (44)$$

5. CONCLUSION

In this work, we apply the HJ formulation approach, using the canonical method, to first-class constrained system with gauge conditions in the framework of the Dirac method. It has been shown that the total Hamiltonian in Dirac's method is obtained in a reduced phase space. This space arises through imposing the gauge fixing conditions in the case of first-class constraints. This leads to elimination of all unknown coefficients from the total Hamiltonian. As a result, the vanishing of these unknown coefficients enabled us to construct the corresponding HJ equation. And, in making use of the same analysis discussed in the canonical method, we have determined the HJF function (S) itself.

Finding S enables us to get the equations of motion using the canonical transformations. This is followed by determining the appropriate wave function in reduced phase space. The constraints then become operators on the wave function to be satisfied in the semi-classical limit.

Following Henneaux *et al.*, (1990), the number of physical degrees of freedom of the Christ-Lee model is found to be one. In our approach, we have shown that the HJF and the wave function are obtained also in terms of one generalized coordinate x . This is in exact agreement with (Henneaux *et al.*, 1990).

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