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# Commutativity Results for Prime and Semi-prime Rings with Derivations

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## ABSTRACT

The main purpose of this paper is to introduce new results concerning prime and semi-prime rings which satisfy some additional conditions; so that we obtain a prime ring that is commutative and a semi-prime ring that contains a non-zero central ideal.


**Keywords:** Prime ring, semi-prime ring, central ideal, derivation and commutative ring.

Mathematical Subject Classification: 16N10, 16W25, 16U80.

## 1- INTRODUCTION

Many studies were done on derivations and commutativity in prime and semi-prime rings, (Herstein, 1978) proved that if R is a prime ring of characteristic (not 2) which admits a non-zero derivation, such that; d (x) d (y) = d(y) d(x) for all x, y Є R, then R is commutative. (H.E. Bell and W.S.Martindale, 1987) proved that if R is a prime ring and U is a non-zero right ideal, and if R admits a non-zero derivation d such that [x,d (x)] is central for all x Є U, then R is commutative. M.N.Daif and H.E. Bell (1992) proved that a semi-prime ring R must be commutative if it admits a derivation d such that (i)d ([x,y]) = [x,y] for all x,y Є R, or (ii) d([x,y])+ [x,y] =0 for all x,y Є R. H.E. Bell and M.N. Daif (1995) proved that; if R is a prime ring and U is a non-zero right ideal,and if R admits a non-zero $U^*$ - derivation d, then either R is commutative or $d^2$ (U) = d (U) d (U) ={0}. Where d is a $U^*$- derivation If d(x)d(y) +d(xy) = d(y)d(x) + d(yx) for all x,y Є U. Hongan (1997) proved that if R is a 2- torsion free semi-prime ring, Z(R) the center of R and d:R→ R is a derivation. If d([x,y]) + [x,y] Є Z(R) or d([x,y]) –[x,y] Є Z (R) for all x, y Є U,U a non-zero ideal of R, then R is commutative. M.N. Daif (1998) proved that; if R is a semi-prime ring and U is a non zero two – sided ideal of R, and R admits a $U^{***}$-derivation d which is non-zero on U, then R contains a non-zero central ideal, where d is a $U^{***}$- derivation if d(xy)= d (yx) for all x,y Є U. In this paper we will prove new results on prime and semi-prime rings with derivations.

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## 2- PRELIMINARIES:

Throughout this paper, R denotes a ring, and it is called a semi-prime ring if a Ra= (0), with a Є R implies a=0, also R is called a prime ring if aRb = (0), a, b Є R, implies that a=0 or b=0. The ring R is said to be n-torsion free, where n ≠ 0 is an integer, if wherever nx = 0, with x Є R, then x =0.

If U is a non-empty subset of R, then the centralizer of U in R, denoted by $C_R$ (U), is defined by: $C_R$ (U)= {a Є R/ ax = xa for all x Є U}. If a Є $C_R$ (U) we say that a centralizes U. An additive map d from R to R is called a derivation if d (xy)= d (x)y + x d(y) for all x,y Є R. We write [x,y] = xy-yx and xoy = xy+ yx. Note the important identities [x,yz] = y[x,y]+ [x,y]z and [xy,z] = x[y,z]+ [x,z]y. A map d is called an inner derivation if there exists a Є R such that d(x) = [a,x] for all x Є R and it is called the derivation induced by a. Let U be a subset of R, a map d:R→R is said to be centralizing on U if [x, d(x)] Є Z(R) for all x Є U, and is said to be skew – centralizing on U if d( x)x + x d(x) Є Z(R) for all x Є U, it is also said to be n-centralizing on S(resp.n-skew-centralizing on S)if $[d(x),x^n]$Є Z(R) for all xЄ S(resp.$d(x)x^n+x^n d(x)$ Є Z(R) for all x Є S).

Now, check the following definitions:

**Definition 1:**

Let R be a prime ring and U a non- zero ideal of R. If d is a non- zero derivation on R such that [d(x) ,y] = d([x,y]) for all x,y Є U. We say that d is a $U^{d1}$- derivation.

**Definition 2:**

Let R be a semi-prime ring and U a non- zero ideal of R. If d is a non- zero derivation on R such that [d(x), d(y)] = [x,y] for all x,y Є U. We say that d is a $U^{d2}$- derivation.

**Definition 3:**

Let R be a prime ring and U a non- zero one-sided ideal of R. If d is a non- zero derivation on R such that d(y)±d(x)= [d(x),y]±[ x, d(y)] for all x,y Є U. We say that d is a $U^{d3}$-derivation.

**Definition 4:**

Let R be a prime ring and U a non-zero one- sided ideal of R. If d is a non- zero derivation on R such that d(y) – [ d(y) ,x]= [ d(x) ,y] –d(x) for all x,y Є U. We say that d is a $U^{d4}$- derivation.

We need to state the following results.

**Lemma 1 [Bresser, 1993]:**

Let R be a prime ring and U is a non-zero left ideal. If R admits a derivation d with d (U) ≠ {0}, satisfying one of the following conditions:

(i) d is centralizing on U.

(ii) d is skew-centralizing on U. Then R is commutative.

**Lemma 2 [Bell and Daif, 1995, Theorem 4]**

Let R be a prime ring and U a non-zero right ideal. If R admits a non- zero derivation d such that [x, d(x)] is central for all x Є U, then R is commutative.

**Lemma 3 [Deng and Bell, 1995, Theorem 2]**

Let n be a fixed integer, let R be n!- torsion free semi-prime ring and U be a non-zero left ideal of R. If R admits a derivation d, which is non-zero on U and n-centralizing on U, then R contains a non-zero central ideal.

**Lemma 4 [Herstein, 1978]:**

If R is a prime ring of characteristic not 2, which admits a non- zero derivation d such that d(x) d(y)= d(y) d(x) for all x,y Є R. Then R is commutative.

**Lemma 5 [Bell and Martindale, 1987, Theorem 3]:**

Let R be a semi-prime ring and U be a non-zero left ideal. If R admits a derivation d, which is non-zero on U and centralizing on U, then R contains a non-zero central ideal.

**Lemma 6 [Hongan, 1997, Lemma 1(1)]:**

Let R be a semi-prime ring and U be a non- zero ideal of R, and let b Є U, if [b,x]=0 for all x Є U, Then b Є Z(R), therefore, if U is commutative then U ⊆ Z(R).

**Lemma 7 [Hongan, 1997, Lemma 1(2)]:**

Let R be a semi-prime ring, U a non-zero ideal of R, a Є R. If [a, x] Є Z(R), for all x Є U, then a Є $C_R$ (U).

**Lemma 8 [Lanski, 1997, Main Theorem]:**

Let R be a semi-prime ring,d a non-zero derivation of R,and U a non-zero left ideal of R.If for some positive integers $t_0,t_1,\ldots,t_n$ and all x Є U, the identity

$[[\ldots[[d(x^{t_0}),x^{t_1}],x^{t_2}],\ldots]x^{t_n}]=0$ holds, then either d(U)=0

or else d(U) and d(R)U are contained in a non-zero central ideal of R.In particular when R is a prime ring, R is commutative.

## 3- THE MAIN RESULTS ON PRIME RINGS

**Theorem 3.1:**

Let R be a prime ring and U a non-zero ideal of R .If R admits a non-zero $U^{d1}$- derivation d, then R is commutative.

**Proof:** Since d is a $U^{d1}$- derivation, we have [d(x), y]= d( [x,y]) for all x,y Є U. Thus, by replacing x for y, we obtain [d(x) ,x] =0 for all x Є U . Hence, by Lemma 1, R is commutative.

Now, we can generalize the above theorem as follows:

**Theorem 3.2:**

Let R be a prime ring and U a non-zero ideal of R. If R admits a non-zero derivation d such that [d(x),y]– d([x,y]) Є Z(R) for all x,y Є U, then R is commutative.

**Proof:** Since we have [ d(x) ,y]- d( [x,y]) Є Z(R ) for all x,y Є U, replacing x by y, we obtain [ d(y) ,y] Є Z( R) for all y Є U. Therefore, by Lemma 2, R is commutative.

## 4. THE MAIN RESULTS ON SEMI-PRIME RINGS

The following theorem is considered as an extension of Lemma 4.

**Theorem 4.1:**

Let R be a 2-torsion free semi-prime ring and U a non- zero ideal of R. If R admits a non-zero $U^{d2}$- derivation d, then R contains a non-zero central ideal.

**Proof:** Since we have d a $U^{d2}$- derivation, we get

[d(x), d(y)] = [x,y], for all x,y Є U. (1)

Then

[d(x), d(y)] –[ x,y]=0 for all x,y Є U.

Replacing y by yz, we obtain [d(x), d(yz)]- [ x,yz]=0 for all x,y,z Є U.

Then

[d(x) ,d(y)z] + [d(x), yd(z)] – y[ x,z] –[x,y]z=0 for all x,y,z Є U.

d(y)[d(x),z]+[d(x),d(y)]z+y[d(x),d(z)] + [ d(x),y] d(z)- y[ x,z]-[ x,y] z=0

for all x,y,z Є U. Then according to (1), we obtain d(y) [d(x),z]+ [ d(x),y] d(z) =0 for all x,y,z Є U. (2)

Replacing z by zd(x), we obtain

d(y) [d(x),zd(x)]+ [ d(x),y] d(zd(x)) =0 for all x,y,z Є U.

Then

d(y) [ d(x),z] d(x)+ [ d(x),y] d(z)d(x)+ [d(x), y]z $d^2$(x)=0 for all x,y,z Є U.

(d(y)[d(x),z]+ [d(x),y]d(z))d(x) + [ d(x),y]z$d^2$(x)=0 for all x,y,z ЄU.

Now according to (2), we get [d(x),y]zd(x)=0 for all x,y,z ЄU.Since U is an ideal,so we have

[d(x),y] zR$d^2$(x) =0 for all x,y,z Є U. (3)

Let $\{P_\alpha : \alpha \in \Lambda\}$ be a family of prime ideals of R, such that $\bigcap_\alpha p_\alpha = \{0\}$. Now (3) yields

$[d(x), y]zR\, d^2(x) = \{0\}$ for all x,y,z Є U.

Hence, for each $P_\alpha$, we either have:

(a) [d(x),y]U $\subseteq P_\alpha$ for all x,y Є U or( b)- d(U)$\subseteq P_\alpha$.

Call $P_\alpha$ an (a)- prime ideal or (b)- prime ideal, according to which one of these conditions is satisfied , note that [ d(x),y] RU$\subseteq P_\alpha$; for each (a)- prime $P_\alpha$, so either [d(x),y] Є $P_\alpha$, for all x,y Є U or U $\subseteq P_\alpha$.

In either event

[d(x),y] Є $P_\alpha$ for all x,y Є U , and all (a)- prime. (4)

Now consider (b)- prime ideals, taking x,y Є U, we have:

$d^2$(xy)= $d^2$(x)y + x $d^2$(y) + 2d(x) d(y) Є $\bigcap_\alpha P_\alpha$. So,

2d(x) d(y) Є $P_\alpha$ for all x,y Є U. Replacing y by ty shows that 2d(x)d(ty) Є $P_\alpha$ for all x,y,t Є U, then 2d(x) d(t)y + 2d(x) td(y) Є $P_\alpha$.

2d(x) td(y)Є $P_\alpha$ for all x,y,t Є U,then 2d(x)rtd(y) Є $P_\alpha$,

2d(x) Rtd(y) $\subseteq P_\alpha$ (5)

and

2d(x) tRd(y) $\subseteq P_\alpha$ for all x,y,t Є U. (6)

It follows that either d(U) $\subseteq P_\alpha$ or 2d(x) y and 2yd(x) Є $P_\alpha$ for all x,y Є U. In either case, 2[ d(x),y] Є $P_\alpha$ for all x,y Є U and

(b)- prime $P_\alpha$ thus for all x,y Є U. (7)

We have ((6) and (7)) that 2[ d(x), y] Є $\bigcap_\alpha P_\alpha = \{0\}$.

And since R is 2-torsion, [ d(x),y]=0 for all x, y Є U.

In particular, [d(x),x] =0 , for all x Є U.

Now by Lemma 5; R contains a non-zero central ideal.

**Remark1:**

In Theorem 4.1,if R admits adrevation d such that [d(x), d(y)]=[x, y] for all x,y Є U. Then R contains a non –zero central ideal.

**Proof:** Let d be a derivation such that [d(x),d(y)]= [x,y] for all x,y Є U. If d=0, then [x,y]=0 for all x, y Є U.

Thus, U is non- zero central ideal.

Now, we suppose that d ≠ 0, by Theorem 4.1, R contains a non- zero central ideal.

**Remark2:**

We notice that 4.1 is not true when U is one –sided ideal. The following example explains the above remark.

**Example:**

Let F be a field and $R=\left\{\begin{pmatrix} g & 0 \\ gh & 0 \end{pmatrix} / g,h \in F\right\}$ is a ring with usual multiplication, $a=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $U=\left\{\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} / g, h \in F\right\}$ be a one-sided ideal of R, and let d be the inner derivation given by d(x)=[a,x] for all x Є U. It is readily verified that [d(x),d(y)]=[x,y] for all x,y Є U. But the conclusion of the theorems not true.

Let $x=\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}$ and $y=\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}$, then

$$d(x)=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}-\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & -g \\ 0 & 0 \end{pmatrix}$$

$$d(y)=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}-\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & -h \\ 0 & 0 \end{pmatrix}$$

$$[d(x),d(y)]=\begin{pmatrix} 0 & -g \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -h \\ 0 & 0 \end{pmatrix}-\begin{pmatrix} 0 & -h \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -g \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$[x,y]=\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}-\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} gh-hg & 0 \\ 0 & 0 \end{pmatrix},$$

since g , h Є F,

$[x,y]=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus [x,y]=[d(x),d(y)]. Let $r=\begin{pmatrix} g & 0 \\ gh & 0 \end{pmatrix}$ Є R, then

$$[r,x]=\begin{pmatrix} g & 0 \\ gh & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}-\begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ gh & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ g^2h & 0 \end{pmatrix}$$

∉ U.

i.e. U is non-central ideal.

**Theorem 4.2:**

Let R be a 2-torsion free semi-prime ring and U a non- zero ideal of R.If R admits a derivation, d, such that d (xoy)± [x,y] Є Z(R) for all x, y Є U, then R contains a non- zero central ideal.

**Proof:**

Let d be a derivation such that d (xoy)+[x,y] Є Z(R) for all x,y Є U. If d=0, then, we have [x,y] Є Z (R), thus from Lemmas 5 and 7, we get U⊆Z(R), so U is a central ideal.

Now, suppose d ≠ 0, then we have d (xoy) + [x,y] Є Z (R) for all x,y Є U, thus d (xoy) + [x,y] commute with any element of R, let w Є R, then, [w, d(xoy) + [ x,y] ]= 0 for all x, y Є U, w Є R. Then

[w, d(xoy)] + [w, [x,y]]=0 for all x,y Є U, w Є R.

Replacing x by y, we obtain 2 [w, d($y^2$)] =0 for all y Є U, w Є R. Since R is 2- torsion free semi-prime, then [w, d ($y^2$)] =0.

Now, replacing w by y, we obtain [ y, d($y^2$)] =0 for all y Є U. Then by Lemma8, U is a non- zero central ideal, i.e. R contains a non-zero central ideal.

We obtain similar results when d(xoy) – [x,y] Є Z(R) for all x, y Є U.

**Theorem 4.3:**

Let R be a 2-torsion free semi-prime ring and U a non-zero ideal of R. If R admits a derivation d, such that d([x,y]) ± (xoy) Є Z(R) for all x,y Є U, then R contains a non- zero central ideal.

**Proof:**

We have d a derivation such that d ([x,y]) + (xoy) Є Z (R) for all x,y Є U. If d=0, then xoy Є Z (R), for all x, y Є U. Replacing x by y, we obtain $2y^2$ Є Z(R), for all y Є U. Then, $2y^2$ commute with any element of R, let r Є R, we have 2 [r, $y^2$] =0, for all y Є U. Since R is 2-torsion free semi-prime, we obtain [r, $y^2$] =0 for all y Є U, r Є R.

Now replacing r by d (y) we get [d(y),$y^2$] =0, for all y Є U. Then by Lemma 3, R contains a non-zero central ideal.

Now, suppose that d ≠0 then we have d( [x,y]) + (xoy)

Є Z(R) for all x,y Є U, replacing x by y, we obtain $2y^2$ Є Z ( R ) for all y Є U, then by the same method in the first part, we get R contains a non-zero central ideal.

We obtain a similar result when d ([x,y]) – (xoy) Є Z (R), for all x,y Є U.

**Theorem 4.4:**

Let R be a 2- torsion free semi-prime ring and U a non-zero ideal of R. If R admits a non-zero derivation d such that d (xoy)± d( [x,y]) Є Z( R) for all x,y Є U, then R contains a non-zero central ideal.

**Proof:** Since d be a non-zero derivation such that, d(xoy)+d([x,y])Є Z(R) for all x,y Є U. Replacing x by y, we obtain 2 $d(y^2)$ Є Z ( R ) for all y Є U.

Now, 2 $d(y^2)$ commute with any element of R, let r Є R, then $2[r,d (y^2)]=0$ for all y Є U, r Є R. Since R is 2-torsion free semi-prime ring and replacing r by y, we get $[y,d (y^2)]=0$ for all y Є U. Then by Lemma 8, R contains a non-zero central ideal. We obtain similar results when d(xoy) – d([x,y]) Є Z ( R ) for all x,y Є U.

**Theorem 4.5:**

Let R be a 2-torsion free semi-prime ring and U a non-zero ideal of R. If R admits a derivation d such that d(xoy) ± (xoy) Є Z(R) for all x,y Є U, then R contains a non- zero central ideal.

**Proof:** We have d is a derivation such that, d (xoy) + (xoy) Є Z(R) for all x,y Є U. If d=0, then we obtain xoy Є Z(R) for all x,y Є U. Replacing x by y, we have $2y^2$ Є Z (R) for all y Є U.

Now, $2y^2$ commute with any element of R, let r Є R , then since R is 2-torsion free and when replacing r by d(y) we obtain $[d(y),y^2 ]=0$ for all y Є U. Then by Lemma 3, we get R contains a central ideal.

Now, suppose that d ≠ 0. For any x,y Є U, we have

d(xoy) + (xoy) Є Z (R) for all x,y Є U, replacing y by x we obtain

$d (2x^2) + 2x^2$ Є Z ( R) for all x Є U.

Now, 2 $(d(x^2) + x^2)$ commute with r Є R, then since R is 2-torsion free and when replacing r by x, we get $[d(x^2), x] =0$ for all x Є U. Then by Lemma 8, we obtain R contains a non-zero central ideal .

We get similar results whenever d(xoy) – (xoy) Є Z (R) for all x,y Є U.

**Theorem 4.6:**

Let R be a 2-torsion free semi-prime ring and U a non-zero ideal of R. If R admits a non- zero derivation d such that d (x)oy + xod(y) Є Z (R) for all x,y Є U, then R contains a non-zero central ideal.

**Proof:** We have d as a non-zero derivation, such that

d (x) oy + xd(y) Є U, then

d(x) y + xd(y) + d(y) x + yd(x) Є Z(R), thus

d (xy) + d(yx) Є Z(R) , for all x,y Є U, then

d(xy+yx) Є Z (R) for all x ,y Є U. Thus, d (xoy) Є Z(R) for all x,y Є U. Replacing x by y, we obtain 2 d $(y^2)$ Є Z (R) for all y Є U.

Now, $2d(y^2)$ commutes with any element of R, let w Є R, then $2[w, d(y^2)] =0$ for all y Є U, w Є R. Since R is 2-torsion free semi-prime ring and replacing w by y, we get $[d ( y^2),y] =0$ for all y Є U. Thus by Lemma8, we have R contains a non-zero central ideal.

**Theorem 4.7:**

Let R be a 2-torsion free semi-prime ring and U a non-zero ideal of R. If R admits a derivation d such that d (x) od(y) ± (xoy) Є Z( R ) for all x,y Є U, then R contains a non-zero central ideal.

**Proof:** Let d be a derivation such that d (x) od(y)+(xoy) Є Z(R) for all x, y Є U. If d =0, then we have xoy Є Z(R) for all x, y Є U, replacing y by x we obtain $2x^2$ Є Z(R). Now, $2x^2$ commutes with any element of R, let r Є R then $2 [r, x^2] =0$ for all x Є U . Since R is 2-torsion free, then $[r,x^2] =0$ for all x Є U, r Є R. Replacing r by d(x) we get $[ d(x) ,x^2] =0$ for all x Є U. Then, by Lemma 3, R contains a non-zero central ideal.

Now, suppose d ≠ 0, then we have d(x)od(y) + ( xoy ) Є Z ( R ) for all x, y Є U. Thus d(x)od(y) + (xoy) commutes with any element of R. Let w Є R, then

[w,d(x)od(y) + (xoy)] =0 for all x, y Є U, w Є R.

Replacing y by x, we obtain

$2[w, d(x)d(x) + x^2 ] =0$ for all x Є U, w Є R .

Since R is 2-torsion free, we get

$[w, d(x) d(x)] + [w,x^2] = 0$ for all x Є U, w Є R.
Replacing w by d(x), we obtain $[d(x),x^2] = 0$ for all x Є U.

Then, by Lemma 3, R contains a non-zero central ideal. We obtain similar results when d(x)od(y) – ( xoy) ЄZ (R ) for all x,y Є U.

**Theorem 4.8:**

Let R be a 2-torsion free semi-prime ring and U a non –zero left ideal. If R admits a non-zero derivation d such that d(x)oy-d(xoy) ∈ Z(R) for all x,y ∈ U. Then R contains a non-zero central ideal.

**Proof:** We have d(x)oy-d(xoy) ∈ Z(R) for all x,y ∈ U,this implies xd(y) +d(y)x ∈ Z(R) for all x, y ∈ U.Then x(xd(y)+d(y)x)=(xd(y)+d(y)x)x for all x,y ∈ U.
$x^2 d(y)+ xd(y)x=xd(y)x+d(y)x^2$ for all x,y ∈ U.
This implies $x^2 d(y)=d(y)x^2$ for all x,y ∈ U.
Thus $[d(y),x^2]=0$ for all x,y ∈ U. Replacing y by x , we obtain $[d(x),x^2]=0 \in Z(R)$ for all x ∈ U, so, by Lemma3, R contains a non-zero central ideal.

**Corollary 4.9:**

Let R be a prime ring and U a non- zero ideal of R .If R admits a non- zero derivation d such that d(x)oy- d( xoy) Є Z (R) for all x,y Є U, then R is commutative.

**Proof:** By assumption, d(x)oy – d(xoy) Є Z (R) for all x,y Є U, then d (x) y+ yd(x) –d(xy+yx) Є Z(R) for all x,y Є U, thus we have
-( xd(y) + d(y)x) Є Z (R) for all x,y Є U. Now, –( xd(y) + d(y) x) commutes with x, so we get x[d(y),x] + [ d(y),x]x =0 for all x,y Є U, we obtain $[d(y) ,x^2] =0$ for all x,y Є U. Replacing y by x, we get $[d(x) ,x^2] =0$ for all x Є U. Therefore, by Lemma 3 , we have R is commutative .

**Theorem 4.10:**

Let R be a 2-torsion free semi-prime ring and U a non-zero one-sided ideal of R. If R admits a non-zero $U^{d3}$- derivation d, then R contains a non-zero central ideal.

**Proof:** Since d is a $U^{d3}$- derivation, then, if we have d(y) – d (x) = [ d(x), y]- [ x, d(y)] for all x, y Є U, then replacing y by x, we obtain 2[ d(x), x] =0 for all x Є U. Since R is 2-torsion free semi-prime, then [d(x), x] =0 for all x Є U. Hence, by Lemma 5, we get R contains a non-zero central ideal.

Now, when we have d(y)+ d(x) = [d(x), y]+ [x,d(y)] for all x,y Є U, then, in the above equation, replacing y by xy, we obtain d(x)y+ xd(y)+d(x)= x[d(x),y]+ [d(x),x]y+ d(x)[x,y]+[x,d(x)]y + x[x, d(y)] for all x,y Є U. Then d(x)y + xd(y)+ d(x) = x([ d(x),y]+ [x,d(y)]) + d(x) [x,y] for all x,y Є U. Replacing y by x, we obtain $d(x^2) = - d(x)$ for all x Є U. Replacing x by –x, we get
$d(x^2) = d(x)$ for all x Є U. (8)

Multiplying (1) from left by y, we obtain
$yd(x^2) = y\, d(x)$ for all x,y Є U. (9)

Multiplying (8) from right by y, we obtain
$d(x^2)\, y = d(x)y$ for all x,y Є U. (10)

By subtracting (10) from (9), it gives
$[d(x^2),y] – [d(x),y] = 0$ for al x,y Є U. (11)

Replacing x by –x , we get
$[d(x^2),y] + [ d(x), y] = 0$ for all x,y Є U. (12)

From (8) we obtain
2[d(x), y] =0 for all x,y Є U. (13)

Since R is 2-torsion free semi-prime ring we get [d(x), y] =0 for all x,y Є U. Replacing y by x using Lemma 5, we get R contains a non-zero central ideal.

**Theorem 4.11:**

Let R be a 2-torsion free semi-prime ring and U a non-zero one-sided ideal of R. If R admits a non-zero $U^{d4}$-derivation d, then R contains a non-zero central ideal.

**Proof:** Since d is a $U^{d4}$-derivation, we have
d(y) + d(x) = [d(x) ,y] –[x, d(y)] for all x,y Є U. (14)

Replacing y by x, we obtain 2d(x) = 2[d (x), x] for all x Є U.
Since R is 2- torsion free semi-prime, then
d(x) = [d(x), x] for all x Є U. (15)

Now, linearization (i.e. putting x-y for x) of (8) gives
d(x)- d(y) = [d(x),x]- [d(y),x] for all x,y Є U.
According to (14) the above calculation reduces to
d(y)= [d(y),x] for all x,y Є U. (16)

Substitution (16) in (14) gives:

d(x) = [d(x), y] for all x,y Є U. (17)

Subtracting (15) from (17) we obtain:

[d(x), y-x] =0 for all x,y Є U. (18)

Putting in (18) 2x for y, it gives:

[d(x),x] =0 for all x Є U. (19)

Then by Lemma 5, we get R contains a non-zero central ideal.

Similarly, we can prove the following corollaries

**Corollary 4.12:**

Let R be a prime ring with char. R ≠ 2 and U a non-zero one–sided ideal of R. If R admits a non-zero $U^{d3}$-derivation d, then R contains a non-zero central ideal.

**Corollary 4.13:**

Let R be a prime ring with char. R ≠ 2 and U a non-zero one- sided ideal of R. If R admits a non-zero $U^{d4}$-derivation d, then R contains a non-zero central ideal.

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# نتائج تبادلية للحلقات الأولية وشبه الأولية مع الاشتقاقات

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## ملخص


الغرض الرئيسي من هذا البحث هو تقديم نتائج جديدة تخص الحلقات الأولية وشبه الأولية، التي تحقق شروطاً اضافية معينة، بحيث تجعل من الحلقة الأولية تبادلية، والحلقة شبه الأولية تحتوي على مثالي مركزي غير صفري.


**الكلمات الدالة:** حلقة أولية، حلقة شبه أولية، مثالي مركزي، اشتقاق، حلقة إبداليـة.

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