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Uniform Quasi-Dedekind Modules

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ABSTRACT

Let R be a commutative ring with 1, and M is a unitary R-module. A submodule N of M is called quasi-invertible if Hom(M/N, M)=0,and M is said to be quasi-Dedekind module if every non-zero submodule of M is quasi-invertible. In this paper, we continue the study of quasi-Dedekind modules that was started by the authors. In particular, we prove that the ring of endomorphisms of a uniform quasi-Dedekind module is an integral domain. We also study quasi-Dedekind modules over Dedekind ring, we prove, among other things, that the only quasi-Dedekind dualizable Z-module is Z. The main result of the paper shows that every uniform faithful quasi-Dedekind R-module is isomorphic to a submodule of Q(R).

KEYWORDS: Dedekind domain, quasi-Dedekind module, ring of endomorphisms, dualizable module, field of quotient.

INTRODUCTION

Let R be a commutative ring with 1, and let M be a unitary (left) R-module. Let N be a submodule of M, following Naoum and Mijbass (in press), we say that N is a quasi-invertible submodule if Hom(M/N, M)=0,and M is said to be quasi-Dedekind module if each non-zero submodule of M is quasi-invertible. In Naoum and Mijbass (in press), the basic properties of quasi-invertible submodules are developed. In Naoum and Mijbass (in press), quasi-Dedekind R-modules are studied. It is proved that an R-module M is quasi-Dedekind iff each non-zero endomorphism is a monomorphism. Moreover, if R is an integral domain and Q(R) is the field of quotients of R, then every R-submodule of Q(R) is uniform quasi-Dedekind module.

In this paper, we continue the study of quasi-Dedekind modules.

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§1: The R-module Q(R)

Let R be an integral domain, and as usual Q(R) is the field of quotients of R. It was shown in Naoum and Mijbass (in press) that every R –submodule of Q(R) is a quasi-Dedekind R-module. In this section, we look at other properties of such kind of modules, in particular, their rings of endomorphisms.

Recall that an R-submodule N of the module M is called invariant submodule if $\forall f \in End(M)$, $f(N) \subseteq N$. We start by the following:

Lemma 1.1:

Let R be an integral domain. The zero R-submodule of Q(R) and Q(R) are the only invariant R-submodules of Q(R).

Proof:

Let N be a non-zero proper R-submodule of Q(R). Since $N \neq Q(R)$, there exists $x \in Q(R)$ and $x \notin N$. Now let $0 \neq b \in N$, define f: $Q(R) \to Q(R)$ as follows: $f(y)=xyb^{-1}$, $\forall y \in Q(R)$. It is clear that f is an R-homomorphism and f(b)=x. Thus $f(N) \not\subset N$ and hence N is not an invariant R-submodule of Q(R).

Proposition 1.2:

Let R be an integral domain. If N is a non-zero R-submodule of Q(R), then $\hat{N}$=Q(R) and either N=Q(R) or $\overline{N}$=QR), where $\hat{N}$ is the injective hull of N and $\overline{N}$ is the quasi-injective hull of N.

Proof:

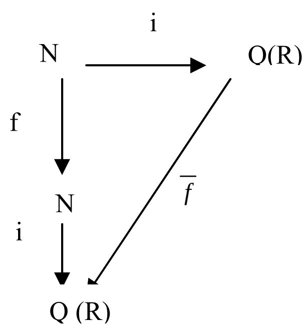
It can be easily seen that N is an essential R-submodule of Q(R), thus $\hat{N}$=Q(R), [Goodearl, 1976, Prop. 1.7, P. 20]. Suppose that N $\neq$ Q(R) and $\overline{N}$ $\neq$ Q(R). By Lemma 1.1, $\overline{N}$ is not an invariant R-submodule of Q(R). This is a contradiction [Goodearl, 1976, Prop. 2.13, P. 48]. Therefore $\overline{N}$=Q(R).

Proposition 1.3:

Let R be an integral domain. If N is an R-submodule of Q(R), then $End_R(N)$ is isomorphic to a subring of the field Q(R), and thus is a commutative ring.

Proof:

The result is trivial if N=(0). Thus we may assume N$\neq$(0). R is an integral domain, thus $\hat{R}$=Q(R), where $\hat{R}$ is the injective hull of R, [Sharpe and Vamos, 1972, Prop. 2.7, p. 34]. It can be easily seen that N is an essential R-submodule of Q(R), thus $\hat{N}$=Q(R), [Goodearl, 1976, Prop. 1.7, P. 20]. Let f$\in$ $End_R(N)$. Since Q(R) is an injective R-module, f can be extended to an R-homomorphism $\overline{f}$: Q(R)$\rightarrow$Q(R) such that $\overline{f}\circ$i=f, where i: N$\rightarrow$Q(R) is the inclusion. That is, the following diagram is commutative.



We claim that $\overline{f}$ is unique. Let $\overline{g}$: Q(R) $\rightarrow$ Q(R) be such that $\overline{g}$ | N= $\overline{f}$ | N. Since Q(R) is a quasi-Dedekind R-module [Naoum and Mijbass, in Press, Examples 1.4(1)], then N is a quasi-invertible R-submodule of Q(R). Thus, since $\overline{g}$ | N= $\overline{f}$ | N, then by [Naoum and Mijbass, in Press, Th. 1.13], $\overline{g}=\overline{f}$. Define φ: End(N)$\rightarrow$End(Q(R)) such that φ(f)= $\overline{f}$. Let f, g$\in$ End(N). φ(g +f)=$\overline{g+f}$. Since $\overline{g}+\overline{f}$ | N=$\overline{g+f}$ | N, then $\overline{g}+\overline{f}=\overline{g+f}$ [Naoum and Mijbass, in Press, Th. 1.13]. Thus φ(g + f)= $\overline{g}+\overline{f}=\varphi$(g)+ φ(f). φ(g$\circ$f)= $\overline{g\circ f}$. Since $\overline{g\circ f}$ | N=$\overline{g}\circ\overline{f}$ | N, then $\overline{g\circ f}=\overline{g}\circ\overline{f}$ [Naoum and Mijbass, in Press, Th.1.13]. Hence φ(g$\circ$f)=$\overline{g}\circ\overline{f}=\varphi$(g)$\circ$ φ(f).Therefore φ is a ring homomorphism. We claim that φ is one-to-one. In fact, let φ(f)=0, thus $\overline{f}$=0. Hence $\overline{f}$ | N= f =0. Therefore End(N) is isomorphic to a subring of End(Q(R)). But End(Q(R)) $\cong$ Q(R), [Kasch, 1982, Lemma 3.7.3, P. 70], then End(N) is isomorphic to a subring of the field Q(R), and End(N) is a commutative ring.

If M and N are submodules of Q(R), we put [N: M]={x$\in$ Q(R) | xM$\subseteq$N}. It is clear that [N: M] is an R-submodule of Q(R).

Proposition 1.4:

Let M and N be R-submodules of Q(R). If M contains R, then $Hom_R(M,N)$ $\cong$[N:M] and hence is a quasi-Dedekind R-module.

Proof:

Let f$\in$ $Hom_R(M,N)$ and f(1)=x. Thus if a/b$\in$ M, then b f(a/b)=f(b. a/b)=f(a)=af(1)= ax, hence f(a/b)=a/b. x. Therefore f is multiplication by x. Now defined $\varphi: Hom_R(M,N)\rightarrow$[N: M] as follows: φ(f)=f(1). It is easy to check that φ is an R-isomorphism. Thus $Hom_R(M,N)$ is R-isomorphic to an R-submodule of Q(R). By [Naoum and Mijbass, in Press, Example

1.4(1)], $Hom_R(M, N)$ is a quasi-Dedekind R-module.

As a special case of Prop.1.4 we have:

Corollary 1.5:

Let L be an R-submodule of Q(R). If L contains R, then $Hom_R(L, L) \cong \{x \in Q(R) \mid x L \subseteq L\}$ and hence is a quasi-Dedekind R-module.

§2: The Rings of Endomorphisms of Uniform Quasi-Dedekind R-Modules

In the last section we studied some aspects of the rings of endomorphisms of submodules of Q(R). In this section we extend these results for arbitrary uniform quasi-Dedekind R-modules. We do this by proving a strong and useful theorem. It shows that every uniform faithful quasi-Dedekind R-module is "essentially" a submodule of Q(R) which contains R.

We start by the following easy proposition. It serves as a motivation for later results.

Proposition 2.1:

If M is a quasi-Dedekind R-module then $End_R(M)$ has no zero divisors.

Proof:

Let f, g$\in End_R(M)$, where f and g are non-zero R-homomorphisms. Thus, there exist m, m′ $\in$ M such that f(m)=x $\neq 0$ and g(m′)=y $\neq 0$, where x, y$\in$ M. By [Naoum and Mijbass, in Press, Th. 1.5], f and g are R-monomorphisms. Hence; $f \circ g(m')=f(y) \neq 0$ and $g \circ f(m)=g(x) \neq 0$. Therefore, $End_R(M)$ has no zero divisors.

We have seen in [Naoum and Mijbass (in Press), Examples 1.4(3)] that if R is an integral domain then Q(R) is a faithful quasi-Dedekind R-module. And we have seen in [Naoum and Mijbass (in Press), Corollary 2.3] that if M is a faithful dualizable R-module, then M is isomorphic to an ideal of R. And this result is false if M is not dualizable (Q, the set of all rational numbers, is not isomorphic to an ideal of Z). The following theorem shows that every faithful quasi-Dedekind R-module is actually a submodule of Q(R). First we need a lemma.

Lemma 2.2:

Let R be an integral domain. If M is a torsion-free uniform R-module, then $S^{-1}M$ is a torsion-free uniform $S^{-1}R$-module for every multiplicative closed subset S of R and hence is an indecomposable $S^{-1}R$-module.

Proof:

It is clear that $S^{-1}M$ is a torsion-free $S^{-1}R$-module. Since M is a torsion-free R-module, then M is a prime R-module. By [Naoum and Mijbass, in Press, Lemma 4.4], $S^{-1}M$ is a uniform $S^{-1}R$-module and thus is an indecomposable $S^{-1}R$-module.

Theorem 2.3:

An R-module M is a uniform faithful quasi-Dedekind R-module if and only if R is an integral domain and M is R-isomorphic to a submodule of Q(R) containing R.

Proof:

Assume that M is a faithful quasi-Dedekind R-module. By [Naoum and Mijbass (in Press), Prop. 1.7], M is a prime R-module. Thus, M is a torsion-free R-module. And (M)=0 is a prime ideal of R, [Naoum and Mijbass (in Press), Corollary 1.8], hence; R is an integral domain. For all x $\neq 0$, x $\in$ M, Rx$\cong$R as R-modules, thus, there exists an R-isomorphism h: $S^{-1}Rx \rightarrow Q(R)= S^{-1}R$, where S=R-{0}. By [Naoum and Mijbass, in Press Prop. 1.7], M is a uniform torsion-free R-module, then $S^{-1}M$ is a uniform torsion-free Q(R)-module (by Lemma 2.2). But Q(R) is a field, hence; $S^{-1}M$ is a vector space over Q(R). Since $S^{-1}M$ is a uniform torsion-free Q(R)-module, then $S^{-1}M$ is a 1-

dimensional Q(R)-vector space, hence; $S^{-1}M \cong Q(R)$ as Q(R)-modules. Since R is an integral domain, then R is a subring of Q(R) and every Q(R)-homomorphism is an R-homomorphism. Thus, there exists an R-isomorphism φ: $S^{-1}M \to Q(R)$. Let $f = h^{-1} \circ \varphi$, then f: $S^{-1}M \to S^{-1}Rx$ is an R-isomorphism. Let $\psi: M \to S^{-1}M$ be the canonical R-homomorphism. Since M is a torsion-free R-module, then ψ is an R-monomorphism. Now $h \circ f \circ \psi : M \to S^{-1}M \to S^{-1}Rx \to Q(R)$ is an R-monomorphism which maps x to 1. Therefore, M is R-isomorphic to a submodule of Q(R) containing R.

The converse, since R is an integral domain, then Q(R) is a quasi-Dedekind R-module [Naoum and Mijbass (in Press), Example 1.4(3)]. By [Naoum and Mijbass (in Press), Corollary 3.15], every R-submodule of Q(R) is a quasi-Dedekind R-module. Thus, M is a uniform faithful quasi-Dedekind R-module.

We are now in a position to state and prove the main result of this section.

Theorem 2.4:

Let M be a uniform quasi-Dedekind R-module and E= End_R (M). Then E is an integral domain and Hom_R (M, M) is a quasi-Dedekind R-module.

Proof:

Put $\overline{R}$ =R/ann(M). Since M is a uniform quasi-Dedekind R-module, then M is a uniform faithful quasi-Dedekind $\overline{R}$-module [Naoum and Mijbass (in Press), Prop. 1.2]. By Th. 2.3, $\overline{R}$ is an integral domain and M is $\overline{R}$-isomorphic to a submodule of Q($\overline{R}$) containing $\overline{R}$. Thus, by Prop. 1.3, $\overline{E} = End_{\overline{R}}$ (M) is an integral domain. By Corollary 1.5, $Hom_{\overline{R}}$ (M, M) is R-isomorphic to an $\overline{R}$-submodule of Q($\overline{R}$). Since Q($\overline{R}$) is a quasi-Dedekind $\overline{R}$-module, thus by [Naoum and Mijbass (in Press), Corollary 3.15], $Hom_{\overline{R}}$ (M, M) is a quasi-Dedekind $\overline{R}$-module. Now since End_R(M)= $End_{\overline{R}}$ (M) (Kasch, 1982), Example (3), P.51, then E is an integral domain. Also since ann_R(M)= ann_R(Hom_R (M, M)) and Hom_R(M, M)= $Hom_{\overline{R}}$ (M, M), (Kasch, 1982), Example (3), P.51, then by [Naoum and Mijbass (in Press), Prop. 1.2], Hom_R (M, M) is a quasi-Dedekind R-module.

Proposition 2.5:

Let R be a Noetherian ring and M is a uniform faithful quasi-Dedekind R-module. If M is a finitely generated R-module, then M is R-isomorphic to an ideal of R.

Proof:

By Th. 2.4, End_R (M) is an integral domain. Since M is finitely generated and End_R (M) is an integral domain, then by [Vasconcelos, 1970, Th. 1.1], M is R-isomorphic to an ideal of R.

It is known that if M is a prime module then $\overline{M}$ is also prime [(Al-Alwan, 1993), Prop. 3.5, Chapter one]. However, $\hat{M}$ may not be prime. The following lemma gives a necessary and sufficient condition for $\hat{M}$ to be a prime module.

Lemma 2.6:

Let M be a prime R-module. Then $\hat{M}$ is a prime R-module if and only if J ($End_R(\hat{M})$)=0.

Proof:

Assume $\hat{M}$ is a prime R-module. Let $f \in J$ ($End_R(\hat{M})$) and $f \neq 0$, then kerf is an essential R-submodule of $\hat{M}$ [Goodearl, 1976, Th. 2.16, P.49]. But this contradicts with [Naoum and Mijbass (in Press), Lemma 3.1].

The converse, let $0 \neq x \in \hat{M}$, then ann($\hat{M}$) $\subseteq$ ann(x). Since M is an essential R-submodule of $\hat{M}$, there exists $r \in R$ such that $0 \neq rx \in M$. Now, let $z \in$ ann(x), then z∈ ann(rx). Since M is a prime module and 0≠ rx∈ M, then z∈ ann(M). Define f: $\hat{M} \to \hat{M}$ as follows: f(m)=zm, for all m∈ $\hat{M}$. It is clear that M⊆ kerf, thus kerf is an essential submodule of $\hat{M}$. Therefore f∈ $J(End_R(\hat{M}))$ [Goodearl, 1976, Th. 2.16, P. 49] and hence f=0. This means $z\hat{M}$=0 and z∈ ann($\hat{M}$). Hence; $\hat{M}$ is a prime R-module.

Let us note that if M is a quasi-Dedekind R-module, End_R (M) may not be a field. Consider the following example.

Example 2.7:

Let Z as a Z-module. Since Z is an integral domain, then Z is a quasi-Dedekind Z-module [Naoum and Mijbass (in Press), Examples 1.4(1)]. End_Z (Z)≅ Z, but Z is not a field.

Proposition 2.8:

If M is a uniform quasi-Dedekind R-module and ann(M)=ann($\hat{M}$), then $End_R(\hat{M})$ is a field.

Proof:

By [Naoum and Mijbass, in Press, Corollary 3.2], $\hat{M}$ is a uniform quasi-Dedekind R-module. By [Naoum and Mijbass (in Press), Prop. 1.7], M and $\hat{M}$ are prime and hence $J(End_R(\hat{M}))$=0 (Lemma 2.6). By [Goodearl, 1976, Th. 2.16, P. 49], $End_R(\hat{M})$ is regular. Since $\hat{M}$ is uniform quasi-Dedekind, then by Th. 3.1 $End_R(\hat{M})$ is an integral domain. Thus $End_R(\hat{M})$ is a regular integral domain and hence is a field.

Corollary 2.9:

Let M be a uniform faithful quasi-Dedekind R-module, then $End_R(\hat{M})$ is a field.

Proof:

Since M is a uniform faithful quasi-Dedekind R-module, then $\hat{M}$ is a uniform faithful quasi-Dedekind R-module [Naoum and Mijbass, in Press, Corollary 3.18]. By Prop. 2.8, $End_R(\hat{M})$ is a field.

Theorem 2.10:

If M is a uniform quasi-Dedekind R-module, then $End_R(\overline{M})$ is a field.

Proof:

By [Naoum and Mijbass (in Press), Corollary 3.16], $\overline{M}$ is a uniform quasi-Dedekind R-module. By Th. 2.4, $End_R(\overline{M})$ is an integral domain. Since M is a quasi-Dedekind R-module, then $J(End_R(\overline{M}))$=0 [Naoum and Mijbass, in Press, Corollary 3.5]. Hence $End_R(\overline{M})$ is a regular ring [Goodearl, 1976, Th. 2.16, P. 49].Thus, $End_R(\overline{M})$ is a regular integral domain, and hence; $End_R(\overline{M})$ is a field.

For an R-module M, there exists an obvious ring monomorphism φ: R/ann(M)→ End_R(M). Thus, one can consider R/ann(M) as a subring of End_R(M).

Proposition 2.11:

If M is a quasi-Dedekind R-module and E=End_R(M), then M is a faithful quasi-Dedekind E-module.

Proof:

Put $\overline{R}$=R/ann(M) and $\overline{E}=End_{\overline{R}}$(M). By [Naoum and Mijbass (in Press), Prop. 1.2], M is a faithful quasi-Dedekind $\overline{R}$-module. Since $\overline{R}$ is embedded in $\overline{E}$, then every $\overline{E}$-homomorphism is $\overline{R}$-homomorphism. Since M is a quasi-Dedekind $\overline{R}$-module, then every non-zero $\overline{E}$-homomorphism is $\overline{E}$-monomorphism [Naoum and Mijbass, in Press, Th. 1.5]. Thus, M is a faithful quasi-Dedekind $\overline{E}$-module. But E=$\overline{E}$ [(Kasch, 1982), Example (3),P. 5.1], hence M is a faithful quasi-Dedekind E-module.

Recall that an R-module M which is finitely generated over $End_R(M)$ is said to be finendo, (Faith, 1972).

Corollary 2.12:

If M is a uniform quasi-Dedekind R-module R-module and ann(M)= Ann($\hat{M}$) and E= $End_R(\hat{M})$, then $\hat{M}$ is a cyclic E-module and hence is finendo.

Proof:

By Prop. 2.8, E is a field. By [Naoum and Mijbass (in Press), Corollary 3.12], $\hat{M}$ is a faithful quasi-Dedekind R-module and hence $\hat{M}$ is a faithful quasi-Dedekind E-module (by Prop. 2.12). Thus, by [Naoum and Mijbass, in Press, Remark 1.3], $\hat{M}$ is an indecomposable E-module. But E is a field, hence $\hat{M} \cong$ E as E-modules.

Corollary 2.13:

Let M be an R-module and E= $End_R(M)$. If M is a uniform quasi-Dedekind R-module, then $\overline{M}$ is a cyclic E-module and hence is finendo.

Proof:

By Th. 2.10, E is a field. And by [Naoum and Mijbass, in Press, Corollary 3.17], $\overline{M}$ is a uniform quasi-Dedekind R-module. Thus, $\overline{M}$ is a faithful quasi-Dedekind E-module (Prop. 2.12) and hence by [Naoum and Mijbass, in Press, Remark 1.3], $\overline{M}$ is an indecomposable E-module. But E is a field, hence $\overline{M} \cong$ E as E-modules.

Proposition 2.14:

Let M and N be uniform quasi-Dedekind R-modules. If ann(M)=ann(N), then $Hom_R(M,N)$ is a quasi-Dedekind R-module.

Proof:

Put $\overline{R}$ =R/ann(M). By [Naoum and Mijbass, in Press, Prop. 1.2] M and N are uniform faithful quasi-Dedekind $\overline{R}$-modules. By Th. 2.3, M and N are $\overline{R}$-isomorphic to submodules A and B of Q($\overline{R}$) that contain $\overline{R}$. Thus $Hom_{\overline{R}}(A,B)$ is a quasi-Dedekind $\overline{R}$-module (by Prop. 1.4). Since $Hom_{\overline{R}}(M,N) \cong Hom_{\overline{R}}(A,B)$, then $Hom_{\overline{R}}(M,N)$ is a quasi-Dedekind $\overline{R}$-module. But $Hom_R(M,N)$ = $Hom_{\overline{R}}(M,N)$, [(Kasch, 1982), Example(3), P.51], thus $Hom_R(M,N)$ is a quasi-Dedekind $\overline{R}$-module. Since ann_R ($Hom_R(M,N)$)= $ann_R(N)$, then by [Naoum and Mijbass, in Press, Prop. 1.2] $Hom_R(M,N)$ is a quasi-Dedekind R-module.

Corollary 2.15:

If M is a uniform faithful quasi-Dedekind R-module, then M^*= $Hom_R(M,R)$ is a quasi-Dedekind R-module.

Proof:

Since M is a faithful quasi-Dedekind R-module, then ann(M)=(0) is a prime ideal of R [Naoum and Mijbass, in Press, Corollary 1.8], and thus R is an integral domain. By [Naoum and Mijbass, in Press, Examples 1.4(1)], R is a faithful quasi-Dedekind R-module. Thus, $Hom_R(M,R)$ is a quasi-Dedekind R-module (Prop. 2.15).

We saw that Z is a quasi-Dedekind Z-module. It is clear that $\overline{Z} = \hat{Z}$ =Q(Z) and Q=Q(Z), where Q is the set of all rational numbers. This fact is true for all uniform faithful quasi-Dedekind R-modules, as the next theorem shows.

Theorem 2.16:

If M is a uniform quasi-Dedekind R-module and $\overline{R}$ =R/ann(M), then $\hat{M} \cong$ Q($\overline{R}$) as $\overline{R}$-modules and either M $\cong$ Q($\overline{R}$) or $\overline{M} \cong$ Q($\overline{R}$) as $\overline{R}$-modules.

Proof:

By [Naoum and Mijbass, in Press, Prop. 1.2], M is a uniform faithful quasi-Dedekind $\overline{R}$-module. Then by Th. 2.3, $\overline{R}$ is an integral domain and M is $\overline{R}$-isomorphic to an $\overline{R}$-submodule L of Q($\overline{R}$) containing $\overline{R}$. Thus $\hat{M} \cong$ Q($\overline{R}$) and either M $\cong$ Q($\overline{R}$) or $\overline{M} \cong$ Q($\overline{R}$) (by Prop. 1.2).

Corollary 2.17:

If M is a uniform faithful quasi-Dedekind R-module, then $\hat{M} \cong Q(R)$ as R-modules and either $M \cong Q(R)$ or $\overline{M} \cong Q(R)$.

The condition on the annihilator in Corollary 2.17 is not superfluous. Consider the following example.

Example 2.18:

Let M= Z_2 as a Z-module. Z_2 is a quasi-Dedekind Z-module and ann(Z_2)= 2Z $\neq$ (0). $\hat{M} = Z_2^{\infty}$ is not isomorphic to Q(Z)=Q.

The converse of Theorem 2.16 is not true as the following example shows.

Example 2.19:

Consider M=Z_4 as a Z_4-module. It is clear that $\hat{M}$ =Q(Z_4)=Z_4. But Z_4 is not a quasi-Dedekind Z_4-module.

In the following theorem we give a condition under which the converse of Theorem 2.16 is true.

Theorem 2.20:

Let M be an R-module and $\overline{R}$=R/ann(M). M is a uniform quasi-Dedekind R-module if and only if ann(M) is a prime ideal of R and $\hat{M} \cong Q(\overline{R})$ as $\overline{R}$-modules.

Proof:

Assume that M is a uniform quasi-Dedekind R-module. By [Naoum and Mijbass, in Press, Corollary 1.8], ann(M) is a prime ideal of R and by Th. 2.17, $\hat{M} \cong Q(\overline{R})$ as $\overline{R}$-modules.

The converse, since ann(M) is a prime ideal of R, then $\overline{R}$ is an integral domain. Thus, Q($\overline{R}$) is a uniform quasi-Dedekind $\overline{R}$-module [Naoum and Mijbass, in Press, Example 1.4(3)]. Thus, $\hat{M}$ is a uniform quasi-Dedekind $\overline{R}$-module. Since M is an $\overline{R}$-submodule of $\hat{M}$, then M is a uniform quasi-Dedekind $\overline{R}$-module [Naoum and Mijbass, on Press, Corollary 3.16]. And by [Naoum and Mijbass, in Press, Prop. 1.2], M is a uniform quasi-Dedekind R-module.

Corollary 2.21:

M is a uniform faithful quasi-Dedekind R-module if and only if R is an integral domain and $\hat{M} \cong Q(R)$ as R-modules.

Proposition 2.22:

Let M be a uniform quasi-Dedekind R-module. If M is a projective R-module, then M is a multiplication R-module.

Proof:

By Th. 2.4, $End_R(M)$ is commutative. And since M is projective, then M is a multiplication R-module [(Naoum, 1991), Prop. 2.1].

§3: Quasi-Dedekind Modules Over Dedekind Domains

Recall that an integral domain R is called a Dedekind domain if every non-zero ideal of R is invertible. It is known that every non-zero prime ideal of a Dedekind domain is maximal (Larsen and McCarthy, 1971).

Our main result of this section states that every dualizable quasi-Dedekind module over a Dedekind domain is a finitely generated faithful projective and a mutiplication module, and the only dualizable quasi-Dedekind Z-module is Z.

Proposition 3.1:

Let R be a Dedekind domain and let M be an R-module. Then M is a finitely generated uniform faithful quasi-Dedekind R-module if and only if M is isomorphic to an ideal of R.

Proof:

Assume that M is a finitely generated uniform faithful quasi- Dedekind R-module. Since R is a Dedekind domain, then R is Noetherian. By Prop. 2.5, M is R-isomorphic to an ideal of R.

The converse, since R is an integral domain, then R is a quasi-Dedekind R-module [Naoum and Mijbass, in Press, Examples 1.4(1)]. By [Naoum and Mijbass, in Press, Examples 1.4(2)], every ideal of R is a quasi-Dedekind R-module. Since R is Dedekind domain, then R is Noetherian. Thus every ideal of R is a finitely generated faithful ideal of R. Since M is isomorphic to an ideal of R, then M is a finitely generated uniform faithful quasi-Dedekind R-module.

Corollary 3.2:

Let R be a Dedekind domain. Then every finitely generated uniform faithful quasi-Dedekind R-module is a projective and a multiplication module.

Proof:

By Prop. 3.1, M is isomorphic to an ideal of R. Since R is a Dedekind domain, then every non-zero ideal of R is invertible and hence is projective, [Naoum and Al-Alwan, 1996), Th. 4.24, P.125]. Therefore, M is projective. By Prop. 2.22, M is a multiplication module.

Theorem 3.3:

Let R be a Dedekind domain and M is an R-module. If M is a uniform faithful quasi-Dedekind R-module, then M is a flat R-module.

Proof:

By [Naoum and Mijbass, in Press, Corollary 3.16], every R-submodule of M is a uniform quasi-Dedekind R-module. And by Corollary 3.2, every finitely generated R-submodule of M is flat, thus by [Rotman, 1979, Corollary 3.49, P.86], M is flat.

If R is not a Dedekind domain, a faithful quasi-Dedekind R-module may not be flat. Consider the following example.

Example 3.4:

Let R=Z[x]. R is an integral domain, but R is not a Dedekind domain. In fact, (x) is a prime ideal of R, but (x) is not a maximal ideal of R. By [Naoum and Mijbass, in Press, Examples 1.4(3)], Q(R) is a quasi-Dedekind R-module and by [Naoum and Mijbass, in Press, Corollary 2.7] N=(1, x/2) is a quasi-Dedekind R-submodule of Q(R) containing R. Suppose that N is a flat R-module. Note that (-x).1+2.x/2=0. Thus, by [Larsen and McCarthy, 1971, Ex. 13(b), P.33] there exist elements f_1, f_2, f_3,…, f_k $\in$ N and elements b_{ji} $\in$ R, i=1, 2, j=1, 2,…,k such that -x b_{j1} +2. b_{j2} =0, j=1, 2,…, k …(1)

and

$$1=\sum_{j=1}^{k} f_j\, b_{j1} \qquad \text{…(2)}$$

From(1),we get x b_{j1} =2 b_{j2} ,and hence b_{j1} =2 l_{j1} and b_{j2} =x γ_{j2} , where j=1,2,…,k and l_{j1}, γ_{j2} $\in$ Z[x]. Since f_j $\in$ N,j=1,2,…,k, then $f_j = h_j + g_j$. x/2=(2 h_j + x g_j) /2, where, h_j , g_j $\in$ Z[x]. Thus1= $\sum_{j=1}^{k}$ [(2 h_j +x g_j)/2].2 l_{j1} = $\sum_{j=1}^{k}$ (2 h_j +x g_j) l_{j1} =2 $\sum_{j=1}^{k}$ $h_j\, l_{j1}$ +x $\sum_{j=1}^{k}$ $g_j\, l_{j1}$ = 2 α_j + x β_j , where $\alpha_j = \sum_{j=1}^{k} h_j\, l_{j1}$, $\beta_j = \sum_{j=1}^{k} g_j\, l_{j1}$. This is impossible because there is no u $\in$ Z[x] $-\{0\}$ such that u(1-(2 α_j + x β_j))=0. Therefore, N is not a flat R-module.

Proposition 3.5:

Let R be a Dedekind domain and M is an R-module such that ann(M) $\neq$ 0. Then M is a quasi-Dedekind R-module if and only if ann(M) is a maximal ideal of R and M $\cong$ R/ann(M) as R-modules.

Proof:

Suppose that M is a quasi-Dedekind R-module. Thus, by [Naoum and Mijbass, in Press, Corollary 1.8], ann(M) is a prime ideal of R and hence ann(M) is a maximal ideal of R, [(Larsen and McCarthy, 1971), corollary 6.17, 136]. Whence R/ann(M) is a field and hence is self-injective. By [Naoum and Mijbass, in Press, Prop. 3.9], $M \cong R/ann(M)$.

The converse follows from [Naoum and Mijbass, in Press, Prop. 3.9].

In the following proposition, we characterize dualizable quasi-Dedekind modules over Dedekind domains.

Proposition 3.6:

Let R be a Dedekind domain and M is an R-module. Then the following statements are equivalent:-

1- M is a dualizable quasi-Dedekind module.
2- M is isomorphic to an ideal of R and hence is a finitely generated projective module.
3- M is a finitely generated faithful multiplication module.
4- M is a dualizable Dedekind module.

Proof:

$(1) \Rightarrow (2)$. By [Naoum and Mijbass, in Press, Corollary 2.3], M is isomorphic to an ideal of R. Since R is a Dedekind domain, then every non-zero ideal of R is invertible and hence is finitely generated and projective [(Rotman, 1979), Th. 4.24, P. 125]. Therefore, M is a finitely generated projective module.

$(2) \Rightarrow (3)$. By [(Smith, 1969), Th. 1], M is a finitely generated faithful multiplication module.

$(3) \Rightarrow (4)$. By [(Smith, 1988), Th. 11], M is projective and hence M is dualizable. Since R is a Dedekind domain, then by [(Al-Alwan, 1993), Th. 4.3, Chapter two], M is a Dedekind module.

$(4) \Rightarrow (1)$. By [Naoum and Mijbass, in Press, Examples 1.4(5)], M is a quasi-Dedekind module.

Corollary 3.7:

Every dualizable quasi-Dedekind Z-module is cyclic, and is isomorphic to Z.

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المقاسات شبه الديديكاندية المنتظمة

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ملخص

لتكن R حلقة تبادلية ذات عنصر أحادي، وليكن M مقاسا وحدويا على R. يقال ان المقاس الجزئي N في M عكوس اذا كان Hom (M/N, M)=0. ويقال ان المقاس M شبه ديديكاندي إذا كان كل مقاس جزئي غير صفري عكوسا. في هذا البحث، نتابع دراسة هذا الصنف من المقاسات التي بداها الباحثان. ونبرهن بصورة خاصة أن حلقة تشاكلات المقاس النظامي شبه الديديكاندي تكون ساحة، كما ندرس المقاسات شبه الديديكاندية على الحلقات الديديكاندية، ونبرهن، بين أمور اخرى، أنه إذا كان M مقاسا شبه ديديكاندي على Z وكان $M^*\neq 0$ فإن $M\approx Z$. النتيجة الرئيسة في البحث تبين أن كل مقاس مخاص نظامي شبه ديديكاندي على R يكون متشاكلا مع مقاس جزئي من Q(R).

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